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Ambipolar and nonambipolar cross-field diffusions

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Abstract
The two-dimensional (2D) steady state of plasma confined radially by an axial magnetic field is studied. An analytical solution of the 2D diffusion equation is found by a variable separation for constant transport coefficients, without the assumption that the flow is ambipolar. The analytical solution is employed for comparing ambipolar flow with the 'short-circuit' limit of a nonambipolar flow. The nonlinear one-dimensional (1D) cross-field diffusion problem is formulated, where the nonlinearity results from the dependence of the transport coefficients on the plasma and neutral densities. The different roles of collisions with plasma particles and with neutral particles are unfolded in the two types of flow, ambipolar and nonambipolar, in either flattening the density profile or making it more peaked. The effect of neutral depletion on the cross-field diffusion is explored.

1. Introduction

Understanding the conditions for the dominance of ambipolarity is one of the many important contributions of Langmuir [1] to the science of what he was the first to call plasma. The flow becomes ambipolar when electric fields that arise in a quasi-neutral plasma make the ion flux and the electron flux across each point at the plasma boundary identical (so that the net current density there is zero). Langmuir discussed ambipolarity in unmagnetized plasmas only. Ambipolarity is often also assumed to dominate magnetized plasmas. If the plasma flow is ambipolar in such plasmas, the impeding of the electron cross-field diffusion results in the impeding of the ion cross-field diffusion as well. The assumption of ambipolar flow across a magnetic field has been challenged by Simon [2], who claims that only ions need to move across field lines, while the magnetized electrons can leave the plasma quickly along field lines, exhibiting a 'short-circuit' effect. Such nonambipolar flows are often considered [3–8], and the question whether the flow across the magnetic field is ambipolar or nonambipolar, is still being debated [9–11]. We do not discuss here what conditions (insulating or conducting walls, for example [3, 4]) might force the flow to become ambipolar or nonambipolar, but we compare and contrast such ambipolar and nonambipolar flows in magnetized plasmas, both linearly and nonlinearly.

Plasma diffusion across a magnetic field has been studied extensively for decades (see [12], for example). Among many theoretical achievements, the diffusion coefficients have been obtained through a kinetic modeling of swarm-like conditions taking into account also time-dependent fields [13–15]. Cross-field diffusion has also been analyzed through a guiding center approach in which nonlocal effects were taken into account [12]. For simplicity, the diffusion coefficients are determined in this paper by the values of the local plasma parameters within a fluid picture. Also, we restrict the analysis to the diffusion regime for both ions and electrons, although ambipolarity can also hold when the motion is not diffusive [16]. Within this simplified picture we are able to mostly analytically unfold how, linearly and nonlinearly, the values and the profiles of the plasma and neutral densities and of the electric potential are determined in the two types of flow, ambipolar and nonambipolar.

We start our analysis by studying the two-dimensional (2D) steady state of a cylindrical plasma that diffuses axially along a confining magnetic field and radially across the field, in which the transport (diffusion and mobility) coefficients are assumed constant. Ambipolar flow and the 'short-circuit' limit of the nonambipolar flow under these conditions have been compared in detail in the past [2, 3]. The new aspect in this part of the paper is the derivation of an analytical solution of the 2D diffusion equation through a variable separation without the assumption of ambipolarity. If the flow is ambipolar, a variable separation is often employed to derive the well-known solution of the resulting 2D linear diffusion equation [17]. Recently the use of a variable separation has been generalized to address the case that the ion inertia is not neglected, both for a magnetized [18] and unmagnetized plasma [19], both flows were assumed ambipolar. Our analysis is therefore a further generalization of the use of a variable separation to derive analytical solutions for the plasma 2D diffusion and for the electric potential distribution for the case that the flow is not necessarily ambipolar. With an appropriate definition of the variables we view this problem as linear. An interesting result is the generalization of what Langmuir was the first to call the plasma balance equation [1] to also apply to plasmas of a nonambipolar flow. Usually the plasma balance equation, which results from particle balance, determines the value of the electron temperature [20–27]. The general plasma balance equations that we derive here determine, if the electric potential distribution is specified, both the electron and the ion temperatures. We use the derived analytical solutions to examine two special cases. In one case our general solution is reduced to the familiar ambipolar-flow solution. In the second case, as mentioned above, the nonambipolar flow exhibits the 'short circuit' effect [2, 3, 5]—only ions diffuse radially across magnetic field lines while electrons diffuse axially along field lines. The general solution obtained via a variable separation is presented in section 2 and the two special ambipolar and nonambipolar solutions are presented in sections 3 and 4, respectively.

From the solution of the linear diffusion equation we turn to examine how nonlinear effects modify the plasma cross-field transport in the two limits of ambipolar diffusion and 'short-circuit' nonambipolar flow. Collisions impede the transport of unmagnetized particles and enhance cross-field transport of magnetized particles. When the flow is ambipolar the cross-field diffusion that is determined by the magnetized electrons is enhanced by collisions, while, in contrast, in the nonambipolar flow the cross-field transport governed by the assumed unmagnetized ions is impeded by collisions. In our fluid model the diffusion coefficients are functions of the local plasma parameters. Neither kinetic effects [13–15] nor nonlocal effects [12] are included. However, we are able to show within this model how the nonuniform plasma parameters, that determine the collisionality, nonlinearly affect the values and the profiles of the plasma and neutral densities. One of the nonlinear effects that we encounter, when we solve for the radial profile of the plasma density, is the dependence of the diffusion coefficient on the sought-after plasma density, a dependence that results from collisions of the diffusing plasma with plasma particles. Other nonlinear effects result from neutral depletion. Neutral depletion has been studied for a long time and has recently attracted a lot of attention because of the higher power and lower gas-pressure employed in gas discharges [28–53]. In our study here of the plasma cross-field transport we extend the theoretical study of neutral depletion to magnetized plasmas. Neutral depletion affects the collisionality with neutrals of the diffusing plasma particles. Neutral depletion also results in a decrease in the rate of ionization. We explore here the intricate role of these nonlinear effects in modifying the balance between ionization and transport and, as a result, the plasma balance equation and the plasma radial density profile.

In section 5 we derive a one-dimensional (1D) equation that relates the plasma density and the neutral density in a plasma in which cross-field diffusion is dominant. We also write expressions for the nonlinear diffusion coefficients for the two cases of ambipolar and nonambipolar flow. In section 6 we solve the nonlinear diffusion equation when the flow is ambipolar. We show that collisions of the diffusing electrons with ions in this ambipolar-flow case tend to flatten the radial density profile while collisions of electrons with neutrals make that profile more peaked at the center of the discharge. In section 7 we solve the nonlinear diffusion equation when the flow is nonambipolar at the 'short-circuit' limit and we show that the role of collisions of the diffusing unmagnetized ions with plasma and with neutrals is reversed. In this nonambipolar flow, ion collisions with electrons make the radial density more peaked, rather than flattened as in the ambipolar flow. This density peaking in the nonambipolar flow has been discovered by Chen [5]. Collisions of ions with neutrals, on the other hand, flatten the radial density profile when the flow is nonambipolar.

2. Ambipolar and nonambipolar 2D linear diffusion

We start by deriving the governing equations for a diffusing plasma. The time-independent continuity equations for the ions and for the electrons are

$$\vec{\nabla}\cdot\vec{\Gamma}_{\rm i} = S, \qquad \vec{\nabla}\cdot\vec{\Gamma}_{\rm e} = S. \tag{1}$$

Here S is the source term due to ionization and $\vec{\Gamma}_{\rm i}$ and $\vec{\Gamma}_{\rm e}$ are the ion and electron flux densities. Since the source of ions and electrons is the same, the net ion flux equals the net electron flux through any closed surface in the plasma; the current is divergence free. The ion and electron flux densities in the diffusion regime are expressed as [17]

$$\begin{aligned}\vec{\Gamma}_{\rm i} &= -\overrightarrow{\overrightarrow{D}}_{\rm i}\vec{\nabla} n - n\overrightarrow{\overrightarrow{\mu}}_{\rm i}\vec{\nabla}\phi,\\ \vec{\Gamma}_{\rm e} &= -\overrightarrow{\overrightarrow{D}}_{\rm e}\vec{\nabla} n + n\overrightarrow{\overrightarrow{\mu}}_{\rm e}\vec{\nabla}\phi,\end{aligned} \tag{2}$$

where n is the density of the quasi-neutral plasma, ϕ the electric potential and $\overrightarrow{\overrightarrow{D}}_{\rm i,e}$ and $\overrightarrow{\overrightarrow{\mu}}_{\rm i,e}$ the diffusion and mobility tensors. The subscripts i and e denote ions and electrons throughout this paper. In cylindrical geometry with no azimuthal dependence, the radial and axial components of the particle flux densities are expressed as

$$\vec{\Gamma}_{\rm i,e} = -\hat{r}D_{\perp{\rm i,e}}\frac{\partial n}{\partial r} - \hat{z}D_{\|{\rm i,e}}\frac{\partial n}{\partial z} \mp \hat{r}n\mu_{\perp{\rm i,e}}\frac{\partial\phi}{\partial r} \mp \hat{z}n\mu_{\|{\rm i,e}}\frac{\partial\phi}{\partial z}, \tag{3}$$

so that the time-independent continuity equations become

$$\begin{aligned}\vec{\nabla}\cdot\vec{\Gamma}_{\rm i,e} &= \frac{1}{r}\frac{\partial}{\partial r}\left(-rD_{\perp{\rm i,e}}\frac{\partial n}{\partial r} \mp rn\mu_{\perp{\rm i,e}}\frac{\partial\phi}{\partial r}\right)\\ &\quad + \frac{\partial}{\partial z}\left(-D_{\|{\rm i,e}}\frac{\partial n}{\partial z} \mp n\mu_{\|{\rm i,e}}\frac{\partial\phi}{\partial z}\right) = \beta N n.\end{aligned} \tag{4}$$

Here $\beta \equiv \langle\sigma_{ion}v_{\rm e}\rangle$ where the brackets $\langle\ \rangle$ denote averaging over the electron velocity distribution function, $\sigma_{\rm ion}$ is the ionization cross section, $v_{\rm e}$ is the electron velocity and N is the neutral density. In both expressions for the flux and the continuity equations, as in similar equations in the rest of this paper, the upper sign refers to ions, while the lower sign to electrons. The diffusion and mobility tensors have also nondiagonal terms that induce azimuthal drifts and the resulting diamagnetic current but they do not contribute to the flux divergence. The total (either ion or electron) particle flux is

$$\Gamma_T = \int_0^a 2\pi r\,dr \int_0^{L/2} \beta N n\,dz, \qquad (5)$$

where a is the radius of the cylindrical plasma, L is the length of the plasma column which is symmetrical with respect to the plane at $z = 0$. We look for a solution through a variable separation as has been done in [18], but here we do not restrict the solution to ambipolar flow. We assume that the coefficients are constant and that plasma density and electric potential are expressed as

$$n(z, r) = n_0 f(r) g(z) \qquad \phi(z, r) = \phi_1(z) + \phi_2(r), \qquad (6)$$

where n_0 is the maximal plasma density, $f(0) = g(0) = 1$, and $g(z) = g(-z)$. The total particle flux is then

$$\Gamma_T = 2\pi\beta N n_0 \int_0^a f(r) r\,dr \int_0^{L/2} g(z)\,dz. \qquad (7)$$

In the standard procedure of solving the equations through variable separation we find that

$$\frac{1}{\tau_{r\mathrm{i,e}}} = \frac{1}{fr}\frac{\partial}{\partial r}\left(-rD_{\perp\mathrm{i,e}}\frac{\partial f}{\partial r} \mp rf\mu_{\perp\mathrm{i,e}}\frac{\partial\phi_2}{\partial r}\right), \qquad (8)$$

$$\frac{1}{\tau_{z\mathrm{i,e}}} = \frac{1}{g}\frac{\partial}{\partial z}\left(-D_{\|\mathrm{i,e}}\frac{\partial g}{\partial z} \mp g\mu_{\|\mathrm{i,e}}\frac{\partial\phi_1}{\partial z}\right) \qquad (9)$$

and

$$\frac{1}{\tau_{r\mathrm{i}}} + \frac{1}{\tau_{z\mathrm{i}}} = \frac{1}{\tau_{r\mathrm{e}}} + \frac{1}{\tau_{z\mathrm{e}}} = \beta N, \qquad (10)$$

where $\tau_{r\mathrm{i}}$, $\tau_{z\mathrm{i}}$, $\tau_{r\mathrm{e}}$ and $\tau_{z\mathrm{e}}$ are constants that express characteristic times the values of which should be determined. These equations are not linear in ϕ_1 and in ϕ_2. However, they are linear in $f\partial\phi_2/\partial r$ and in $g\partial\phi_1/\partial z$, and we, therefore, call this case of specified and constant transport coefficients the linear case. It is easy to verify that

$$f(r) = J_0(pr/a) \qquad g(z) = \cos(\pi z/L),$$
$$\phi_1(z) = \phi_{1,0}\ln\cos(\pi z/L) \qquad \phi_2(r) = \phi_{2,0}\ln J_0(pr/a), \qquad (11)$$

in which $\phi_{1,0}$ and $\phi_{2,0}$ are constants, are solutions of equations (8) and (9), so that

$$\frac{1}{\tau_{r\mathrm{i,e}}} = \frac{p^2\left(D_{\perp\mathrm{i,e}} \pm \mu_{\perp\mathrm{i,e}}\phi_{2,0}\right)}{a^2}; \qquad \frac{1}{\tau_{z\mathrm{i,e}}} = \frac{\pi^2\left(D_{\|\mathrm{i,e}} \pm \mu_{\|\mathrm{i,e}}\phi_{1,0}\right)}{L^2}. \qquad (12)$$

Here $p = 2.4048$ is the first zero of the Bessel function J_0. The density is zero at the boundaries at $r = a$ and at $z = \pm L/2$. The electric field and the plasma potential are infinite at the boundaries. We discuss later how the boundary conditions should be modified in order to find the finite potential drop across the plasma.

We also note that if we define a Boltzmann density as

$$n_B = n_0\exp(e\phi/T_e) = n_0[J_0(pr/a)]^{e\phi_{2,0}/T_e}[\cos(\pi z/L)]^{e\phi_{1,0}/T_e}, \qquad (13)$$

in which T_e is the (assumed uniform) electron temperature and e is the elementary charge, the ratio of the plasma density to the Boltzmann density becomes

$$\frac{n}{n_B} = [J_0(pr/a)]^{1-e\phi_{2,0}/T_e}[\cos(\pi z/L)]^{1-e\phi_{1,0}/T_e}. \qquad (14)$$

The plasma density is reduced to the Boltzmann density when $\phi_{1,0} = \phi_{2,0} = T_e/e$.

Employing equations (10) and (12) that express particle balance we therefore write

$$\frac{p^2\left(D_{\perp\mathrm{i}} + \mu_{\perp\mathrm{i}}\phi_{2,0}\right)}{a^2} + \frac{\pi^2\left(D_{\|\mathrm{i}} + \mu_{\|\mathrm{i}}\phi_{1,0}\right)}{L^2} = \beta N. \qquad (15)$$

$$\frac{p^2\left(D_{\perp\mathrm{e}} - \mu_{\perp\mathrm{e}}\phi_{2,0}\right)}{a^2} + \frac{\pi^2\left(D_{\|\mathrm{e}} - \mu_{\|\mathrm{e}}\phi_{1,0}\right)}{L^2} = \beta N. \qquad (16)$$

These equations are a generalization of the plasma balance equation, as introduced by Langmuir [1], to the case that ambipolarity is not imposed. Since when the flow is not necessarily ambipolar $\phi_{1,0}$, $\phi_{2,0}$ are not known, we need to make further assumptions in order to be able to solve the equations. If the boundary conditions are such that $\phi_{1,0}$ and $\phi_{2,0}$ are specified, equations (15) and (16) can be viewed as determining both electron and ion temperatures. Usually, when ambipolarity is assumed, it is the ambipolar electric field and the electron temperature that are viewed as determined by the solution of the plasma balance equation [17–27].

Before solving the equations we write expressions for the particle flux densities and velocities and for the electric potential. Note that at the axial boundaries the flux has an axial component only while at the radial boundaries the flux has a radial component only. The total axial flux is

$$\Gamma_{zT\mathrm{i,e}} = \hat{z}\cdot\int_0^a 2\pi r\vec{\Gamma}_{\mathrm{i,e}}(z = \pm L/2, r)\,dr = (2/p)J_1(p)La^2 n_0/\tau_{z\mathrm{i,e}}, \qquad (17)$$

the total radial flux is

$$\Gamma_{rT\mathrm{i,e}} = \hat{r}\cdot\int_0^{L/2} 2\pi a\vec{\Gamma}_{\mathrm{i,e}}(z, r = a)\,dz = (2/p)J_1(p)La^2 n_0/\tau_{r\mathrm{i,e}} \qquad (18)$$

and the total particle flux is

$$\Gamma_T = \Gamma_{zT\mathrm{i,e}} + \Gamma_{rT\mathrm{i,e}} = (2/p)\,J_1(p)La^2\beta N n_0. \qquad (19)$$

Here J_1 is the Bessel function. The expression for the total flux (19) is equivalently obtained by performing the integral in (7) into which the expressions (11) are substituted.

The total number of plasma particles is

$$n_T = 2\pi n_0\int_0^a f(r) r\,dr\int_0^{L/2} g(z)\,dz = (2/p)J_1(p)La^2 n_0. \qquad (20)$$

We find that the characteristic times defined above are characteristic residence or diffusion times. They denote how much time, for given fluxes and number of particles, it takes for all ions or electrons to exit through the boundaries either radially ($\tau_{r\mathrm{i,e}}$) or axially ($\tau_{z\mathrm{i,e}}$):

$$\frac{n_T}{\Gamma_{rT\mathrm{i,e}}} = \tau_{r\mathrm{i,e}}; \qquad \frac{n_T}{\Gamma_{zT\mathrm{i,e}}} = \tau_{z\mathrm{i,e}}. \qquad (21)$$

By defining

$$\tau_{\rm T} \equiv \frac{n_{\rm T}}{\Gamma_{\rm T}}, \tag{22}$$

we can write

$$\frac{1}{\tau_{\rm T}} = \frac{1}{\tau_{r{\rm i,e}}} + \frac{1}{\tau_{z{\rm i,e}}} = \frac{\Gamma_{\rm T}}{n_{\rm T}} = \frac{\Gamma_{rT{\rm i,e}} + \Gamma_{zT{\rm i,e}}}{n_{\rm T}} = \beta N. \tag{23}$$

With the expressions for the particle flux densities and densities we express the particle velocities as

$$\vec{v}_{\rm i,e} = \frac{\vec{\Gamma}_{\rm i,e}}{n} = \hat{r}\frac{aJ_1(pr/a)}{\tau_{r{\rm i,e}}p^2 J_0(pr/a)} + \hat{z}\frac{L\tan(\pi z/L)}{\tau_{z{\rm i,e}}\pi^2}. \tag{24}$$

The velocities at the boundaries are infinite within the diffusion model. A relation between the axial velocity and the potential follows (11) and (24):

$$\phi_1(z) = -\phi_{1,0}\ln\left[\frac{\pi^2 v_{z{\rm i,e}}(z)\tau_{z{\rm i,e}}}{L}\right]. \tag{25}$$

The relation between the radial velocity and the radial potential drop cannot be written analytically. However, at the neighborhood of the plasma boundary ($r = a$) we can approximate $J_0(pr/a) \cong -pJ_1(p)(r/a-1)$ so that

$$v_{r{\rm i,e}}(r) \cong \frac{a}{\tau_{r{\rm i,e}}p^3(1-r/a)}, \qquad r \cong a. \tag{26}$$

Employing equation (11) we express the potential near the boundary as

$$\phi_2(r) = \phi_{2,0}\ln[pJ_1(p)(1-r/a)], \qquad r \cong a.$$

Using the last two expressions, we write

$$\phi_2(r) = -\phi_{2,0}\ln\left[\frac{p^2 v_{r{\rm i,e}}(r)\tau_{r{\rm i,e}}}{aJ_1(p)}\right], \qquad r \cong a. \tag{27}$$

Once finite values are specified for the velocities at the boundaries, finite values for the potential drop across the plasma are determined by (25) and (27).

In the next section we recover ambipolar diffusion from the equations described above.

3. Linear ambipolar diffusion

We assume ambipolarity such that the ion and the electron flux densities are identical not only at the plasma boundary, but their radial and axial components are also equal at each location in the plasma:

$$\vec{\Gamma}_{\rm i} - \hat{\varphi}\cdot\vec{\Gamma}_{\rm i} = \vec{\Gamma}_{\rm e} - \hat{\varphi}\cdot\vec{\Gamma}_{\rm e}. \tag{28}$$

As is mentioned above, the ion and electron flux densities have different azimuthal components that result in an azimuthal diamagnetic current. From the imposed equality of the electron and ion radial and axial flux densities we obtain that

$$\frac{1}{\tau_r} \equiv \frac{1}{\tau_{r{\rm i}}} = \frac{p^2(D_{\perp{\rm i}} + \mu_{\perp{\rm i}}\phi_{2,0})}{a^2} = \frac{1}{\tau_{r{\rm e}}} = \frac{p^2(D_{\perp{\rm e}} - \mu_{\perp{\rm e}}\phi_{2,0})}{a^2} \tag{29}$$

and

$$\frac{1}{\tau_z} \equiv \frac{1}{\tau_{z{\rm i}}} = \frac{\pi^2\left(D_{\|{\rm i}} + \mu_{\|{\rm i}}\phi_{1,0}\right)}{L^2} = \frac{1}{\tau_{z{\rm e}}} = \frac{\pi^2\left(D_{\|{\rm e}} - \mu_{\|{\rm e}}\phi_{1,0}\right)}{L^2}. \tag{30}$$

These equalities determine the (ambipolar) electric field through the determination of the values of $\phi_{1,0}$ and of $\phi_{2,0}$:

$$\phi_{2,0} = \frac{D_{\perp{\rm e}} - D_{\perp{\rm i}}}{\mu_{\perp{\rm i}} + \mu_{\perp{\rm e}}}, \qquad \phi_{1,0} = \frac{D_{\|{\rm e}} - D_{\|{\rm i}}}{\mu_{\|{\rm i}} + \mu_{\|{\rm e}}}. \tag{31}$$

We may now write

$$\frac{1}{\tau_z} = \frac{\pi^2 D_{\|{\rm a}}}{L^2}, \qquad \frac{1}{\tau_r} = \frac{p^2 D_{\perp{\rm a}}}{a^2},$$
$$D_{\|{\rm a}} \equiv \frac{D_{\|{\rm i}}\mu_{\|{\rm e}} + D_{\|{\rm e}}\mu_{\|{\rm i}}}{\mu_{\|{\rm i}} + \mu_{\|{\rm e}}}, \qquad D_{\perp{\rm a}} \equiv \frac{D_{\perp{\rm i}}\mu_{\perp,{\rm e}} + D_{\perp{\rm e}}\mu_{\perp{\rm i}}}{\mu_{\perp{\rm i}} + \mu_{\perp{\rm e}}}. \tag{32}$$

Here $D_{\|{\rm a}}$ and $D_{\perp{\rm a}}$ are the standard ambipolar diffusion coefficients [17]. The plasma balance equation is reduced to its familiar form [17]:

$$\frac{p^2 D_{\perp{\rm a}}}{a^2} + \frac{\pi^2 D_{\|{\rm a}}}{L^2} = \beta N. \tag{33}$$

The identical ion and electron flux densities are

$$\Gamma_r = D_{\perp{\rm a}}\frac{p}{a}n_0 J_1\left(\frac{pr}{a}\right)\cos(\pi z/L),$$
$$\Gamma_z = D_{\|{\rm a}}\frac{\pi}{L}n_0 J_0\left(\frac{pr}{a}\right)\sin(\pi z/L). \tag{34}$$

We can now modify the model so that the plasma velocity and the electric potential are finite at the boundary. Even though the ion inertia is neglected in our diffusion model, we assume that in the case of ambipolar diffusion the ion and electron velocities at the boundary equal the ion acoustic velocity:

$$v_{r{\rm i,e}} = v_{z{\rm i,e}} = c \equiv \sqrt{\frac{T_{\rm e}}{m_{\rm i}}}. \tag{35}$$

Here $m_{\rm i}$ is the ion mass. Employing equations (25), (27), (31), (35) and the expressions for $\tau_{r{\rm i,e}}$ and for $\tau_{z{\rm i,e}}$, we find that the potential drop across the plasma is

$$\phi_1(z = \pm L/2) = -\left(\frac{D_{\|{\rm e}} - D_{\|{\rm i}}}{\mu_{\|{\rm i}} + \mu_{\|{\rm e}}}\right)\ln\left(\frac{cL}{D_{\|{\rm a}}}\right) \tag{36}$$

and

$$\phi_2(r = a) = -\left(\frac{D_{\perp{\rm e}} - D_{\perp{\rm i}}}{\mu_{\perp{\rm i}} + \mu_{\perp{\rm e}}}\right)\ln\left[\frac{ca}{J_1(p)D_{\perp{\rm a}}}\right]. \tag{37}$$

We now write the explicit expressions for the diffusion and mobility coefficients. Using the Einstein relations we express the coefficients as

$$D_{\|{\rm i,e}} = \frac{T_{\rm i,e}}{e}\mu_{\|{\rm i,e}} = \frac{T_{\rm i,e}}{m_{\rm i,e}\nu_{\rm i,e}}, \qquad D_{\perp{\rm e}} = \frac{T_{\rm e}}{e}\mu_{\perp{\rm e}} = \frac{T_{\rm e}\nu_{\rm e}}{m_{\rm e}\omega_{\rm e}^2},$$
$$D_{\perp{\rm i}} = \frac{T_{\rm i}}{e}\mu_{\perp{\rm i}} = \frac{T_{\rm i}}{m_{\rm i}\nu_{\rm i}\left(1 + \omega_{\rm i}^2/\nu_{\rm i}^2\right)}. \tag{38}$$

Here $\nu_{\rm i}$ and $\nu_{\rm e}$ are the ion and electron collision frequencies, $\omega_{\rm i}$ and $\omega_{\rm e}$ are the ion and electron cyclotron frequencies and

m_e is the electron mass. We assume that $\omega_e \gg \nu_e$, the electrons are assumed magnetized, but we keep the form of the ion coefficients, $D_{\perp i}$ and $\mu_{\perp i}$, general, so that this form describes both magnetized and unmagnetized ions. Employing the ordering

$$D_{\|e} \gg D_{\|i}, \qquad \mu_{\|e} \gg \mu_{\|i}, \qquad D_{\perp i} \gg D_{\perp e},$$
$$\mu_{\perp i} \gg \mu_{\perp e}, \qquad T_e \gg T_i, \tag{39}$$

we obtain the approximate expressions (as, for example, in [17])

$$\phi_{2,0} = -\frac{T_i}{e}, \qquad \phi_{1,0} = \frac{T_e}{e}, \qquad D_{\|a} = \frac{D_{\|,e}\mu_{\|,i}}{\mu_{\|,e}} = \frac{T_e}{m_i \nu_i},$$
$$D_{\perp a} = D_{\perp e}. \tag{40}$$

The particle flux densities are approximately

$$\Gamma_r = \frac{T_e \nu_e}{m_e \omega_e^2} \frac{p}{a} n_0 J_1\left(\frac{pr}{a}\right) \cos(\pi z/L),$$
$$\Gamma_z = \frac{c^2}{\nu_i} \frac{\pi}{L} n_0 J_0\left(\frac{pr}{a}\right) \sin(\pi z/L). \tag{41}$$

The plasma potentials are

$$\phi_1(z = \pm L/2) = -\frac{T_e}{e} \ln\left(\frac{L\nu_i}{c}\right),$$
$$\phi_2(r = a) = \frac{T_i}{e} \ln\left[\frac{cam_e\omega_e^2}{J_1(p)T_e\nu_e}\right]. \tag{42}$$

The ambipolar radial electric field is negative, pointing towards the center of the discharge. It is proportional to the ion temperature.

We end up with the plasma balance equation:

$$\frac{p^2}{a^2}\frac{T_e\nu_e}{m_e\omega_e^2} + \frac{\pi^2}{L^2}\frac{T_e}{m_i\nu_i} = \beta N. \tag{43}$$

When the flow is ambipolar the ion temperature, smaller than the electron temperature, does not show up in the equation. The electron temperature turns out to be determined by the plasma balance equation [1]. The ion temperature is not determined by the plasma balance equation. Particle balance is not sufficient to determine both electron and ion temperatures.

4. Linear nonambipolar diffusion

When the flow is not ambipolar various different flows are possible with associated different potential distributions, as described by (15) and (16). Let us assume, as in [3], that the radial electric field is zero,

$$\phi_{2,0} = 0, \tag{44}$$

but that an axial electric field might exist. The characteristic times become

$$\frac{1}{\tau_{ri,e}} = \frac{p^2 D_{\perp i,e}}{a^2}, \qquad \frac{1}{\tau_{zi,e}} = \frac{\pi^2\left(D_{\|i,e} \pm \mu_{\|i,e}\phi_{1,0}\right)}{L^2}, \tag{45}$$

so that the plasma balance equations take the form

$$\frac{1}{\tau_{ri}} + \frac{1}{\tau_{zi}} = \frac{p^2 D_{\perp i}}{a^2} + \frac{\pi^2\left(D_{\|i} + \mu_{\|i}\phi_{1,0}\right)}{L^2} = \beta N \tag{46}$$

for the ions and

$$\frac{1}{\tau_{re}} + \frac{1}{\tau_{ze}} = \frac{p^2 D_{\perp e}}{a^2} + \frac{\pi^2\left(D_{\|e} - \mu_{\|e}\phi_{1,0}\right)}{L^2} = \beta N \tag{47}$$

for the electrons. These equations determine the axial potential amplitude as

$$\phi_{1,0} = \frac{(D_{\perp e} - D_{\perp i})}{\left(\mu_{\|i} + \mu_{\|e}\right)}\frac{p^2L^2}{\pi^2a^2} + \frac{\left(D_{\|e} - D_{\|i}\right)}{\left(\mu_{\|i} + \mu_{\|e}\right)}. \tag{48}$$

One of the plasma balance equations ((46) or (47)) and the expression for the electric potential (48) can be viewed as the governing equations. At the 'short-circuit' limit [2, 3], the plasma length is larger than the plasma radius so that ions move mostly radially, but the high-mobility electrons move mostly axially along field lines

$$\frac{1}{\tau_{zi}} \ll \frac{1}{\tau_{ri}}, \qquad \frac{1}{\tau_{re}} \ll \frac{1}{\tau_{ze}}. \tag{49}$$

These relations are obtained from expressions (39). The equations are reduced to

$$\frac{p^2 D_{\perp i}}{a^2} = \beta N, \qquad \frac{\pi^2\left(D_{\|e} - \mu_{\|e}\phi_{1,0}\right)}{L^2} = \beta N,$$
$$\phi_{1,0} = -\frac{D_{\perp,i}}{\mu_{\|,e}}\frac{p^2L^2}{\pi^2a^2} + \frac{T_e}{e}. \tag{50}$$

We choose as governing equations the following two equations:

$$\frac{p^2 T_i}{a^2 m_i \nu_i\left(1 + \omega_i^2/\nu_i^2\right)} = \beta N,$$
$$\phi_{1,0} = -\frac{T_i m_e \nu_e}{e m_i \nu_i\left(1 + \omega_i^2/\nu_i^2\right)}\frac{p^2L^2}{\pi^2a^2} + \frac{T_e}{e}. \tag{51}$$

The first of these two equations relates the ion and the electron temperatures, while the second equation determines the electric potential. Note that the electric potential drop might be either positive or negative.

The ion and electron flux densities are written as

$$\vec{\Gamma}_i = \hat{r}\frac{\beta N a}{p} n_0 J_1\left(\frac{pr}{a}\right)\cos(\pi z/L),$$
$$\vec{\Gamma}_e = \hat{z}\frac{\beta N L}{\pi} n_0 J_0\left(\frac{pr}{a}\right)\sin(\pi z/L), \tag{52}$$

while the potential distribution is

$$\phi(z, r) = \left(-\frac{m_e\nu_e L^2 \beta N}{e\pi^2} + \frac{T_e}{e}\right)\ln\left[\cos(\pi z/L)\right]. \tag{53}$$

The calculation of the finite value of the potential drop across the plasma requires a further study.

A special case is of an identically zero electric field, obtained when $\phi_{1,0} = 0$ is imposed in (48). Then

$$\phi_{1,0} = \phi_{2,0} = 0. \tag{54}$$

At the 'short-circuit' limit, the plasma balance equations for the ions and for the electrons are reduced in that case to

$$\frac{p^2 T_i}{a^2 m_i \nu_i\left(1 + \omega_i^2/\nu_i^2\right)} = \beta N, \qquad \frac{\pi^2 T_e}{L^2 m_e \nu_e} = \beta N. \tag{55}$$

These relations can be viewed as equations that determine both the electron and the ion temperatures, in contrast to the case of ambipolar flow, in which there is a single plasma balance equation (33) that does not determine the ion temperature. The electron temperature is determined by the axial electron dynamics (the second equation in (55)), followed by the ion temperature being determined by the radial ion dynamics (the first equation in (55)) with the already determined electron temperature.

When the ions are unmagnetized and the first of equations (55) becomes $p^2 T_i/(a^2 m_i \nu_i) = \beta N$, the intensity of the magnetic field affects neither the electron nor the ion temperatures. However, when the ions are magnetized, $p^2 T_i \nu_i/(a^2 m_i \omega_i^2) = \beta N$, an increase in the magnetic field intensity does increase the ion temperature, although it still does not affect the electron temperature.

However, the assumption of an identically zero electric field (54) is not realistic. Usually electric fields are expected to arise, as previously described.

In the next section we move to the second part of the paper. We examine the nonlinear cross-field diffusion.

5. Nonlinear cross-field diffusion

We examine cases in which the plasma dynamics is governed by radial cross-field transport, so that

$$\Gamma_{ir} = -D_\perp \frac{\partial n}{\partial r}. \tag{56}$$

There are two cases of interest in which the flux is described this way. The first case is that of ambipolar diffusion, in which

$$D_\perp = D_{\perp e} = \frac{T_e \nu_e}{m_e \omega_e^2}, \tag{57}$$

and in which the plasma column is long enough so that we can neglect the axial transport. The electron collision frequency is

$$\nu_e = k_{ei} n + k_{eN} N. \tag{58}$$

The electrons collide with ions and with neutrals, where the relative velocity is the electron azimuthal drift velocity. Here k_{ei} and k_{eN} are the electron–ion and the electron–neutral collision rate constants. The second case is the 'short-circuit' limit of the nonambipolar flow in which ions are unmagnetized and the radial electric field is zero, as described in the previous section. Then

$$D_\perp = D_{\perp i} = \frac{T_i}{m_i \nu_i}. \tag{59}$$

The ion collision frequency in this case is [5]

$$\nu_i = k_{ie} n + k_{iN} N, \tag{60}$$

where the unmagnetized ions are slowed by collisions with the magnetized electrons and with neutrals, and k_{ie} and k_{iN} are the ion–electron and ion–neutral collision rate constants. We take the axially averaged densities in the expressions for the collision frequencies. The dynamics of the electrons and of the ions is therefore one dimensional.

For the analysis we express the diffusion coefficients as

$$D_{\perp e} \equiv D_{\perp ei}\bar{n} + D_{\perp eN}\bar{N}, \qquad D_{\perp i} \equiv D_{\perp ie}\bar{n} + D_{\perp iN}\bar{N}, \tag{61}$$

where

$$D_{\perp ei} \equiv \frac{T_e k_{ei} n_0}{m_e \omega_e^2}, \qquad D_{\perp eN} \equiv \frac{T_e k_{eN} N_W}{m_e \omega_e^2},$$
$$D_{\perp ie} \equiv \frac{T_i}{m_i k_{ie} n_0}, \qquad D_{\perp iN} \equiv \frac{T_i}{m_i k_{iN} N_W}, \tag{62}$$

and

$$\bar{n} \equiv \frac{n}{n_0}, \qquad \bar{N} \equiv \frac{N}{N_W}. \tag{63}$$

Here N_W is the neutral-gas density at the radial boundary.

We turn to the neutral dynamics. We assume here that the pressure gradient of the neutrals is balanced by the drag force exerted by the ions, as in [40] and in [42]. We therefore write

$$\frac{\partial (N T_g)}{\partial r} = m_i k_{iN} N \Gamma_{ir}, \tag{64}$$

where T_g is the (assumed uniform) neutral-gas temperature. We do not take the ion mobility as variable as done by Godyak [25]. Also, the drag force is proportional to the ion radial flux because we assume that

$$v_i \gg v_N, \tag{65}$$

where v_N is the neutral radial drift velocity. Note that the neutrals and the plasma are not in pressure balance, as they are in [40] and in [42]. There is a pressure balance between plasma, neutral and magnetic field pressure. Combining (56) and (64), we obtain

$$\frac{D_N}{\bar{N}}\frac{\partial N}{\partial r} + D_\perp \frac{\partial n}{\partial r} = 0, \tag{66}$$

where

$$D_N \equiv \frac{T_g}{m_i k_{iN} N_W}. \tag{67}$$

If $D_\perp = D_\perp(N, n)$ we can write this equation as

$$\frac{\partial \bar{n}}{\partial \bar{N}} = -\frac{N_W D_N}{n_0 D_\perp (N, n) \bar{N}}. \tag{68}$$

We are also interested in examining two quantities for each case. The first is the ratio of plasma flux at the boundary to the maximal plasma density

$$c f_s \equiv \frac{\Gamma_{ir}(a)}{n_0} = -\frac{D_\perp (a)}{n_0}\frac{\partial n}{\partial r}(a). \tag{69}$$

Here f_s (h_l in [25] and g_s in [18]) denotes the ratio of the plasma density near the edge (where the ion velocity is the acoustic velocity c) to the maximal density n_0. A second quantity of interest is the rate of neutral depletion, defined in [50] as

$$d \equiv \frac{N_W}{N(0)} - 1 = \frac{1}{\bar{N}(0)} - 1. \tag{70}$$

5.1. Relation between neutral and plasma densities—ambipolar diffusion

For the ambipolar diffusion we employ (57) and (58) to obtain

$$\frac{\partial \overline{N}}{\partial \overline{n}} = -\left(\overline{D_{\perp ei}}\,\overline{n} + \overline{D_{\perp eN}}\,\overline{N}\right)\overline{N}, \tag{71}$$

where

$$\overline{D_{\perp ei}} \equiv \frac{D_{\perp ei} n_0}{D_N N_W} = \frac{T_e m_i k_{iN} k_{ei} n_0^2}{T_g m_e \omega_e^2}, \qquad \overline{D_{\perp eN}} \equiv \frac{D_{\perp eN} n_0}{D_N N_W} = \frac{T_e m_i k_{iN} k_{eN} n_0 N_W}{T_g m_e \omega_e^2}. \tag{72}$$

The solution of this equation is

$$\overline{N} = \frac{\exp[-(\overline{D_{\perp ei}}/2)\overline{n}^2]}{\left[1 + \overline{D_{\perp eN}}\sqrt{\pi/(2\overline{D_{\perp ei}})}\,\mathrm{erf}\left(\sqrt{\overline{D_{\perp ei}}/2}\,\overline{n}\right)\right]}, \tag{73}$$

where erf is the Error function. The rate of neutral depletion is

$$d = \left[1 + \overline{D_{\perp eN}}\sqrt{\pi/(2\overline{D_{\perp ei}})}\,\mathrm{erf}\left(\sqrt{\overline{D_{\perp ei}}/2}\right)\right]\exp(\overline{D_{\perp ei}}/2) - 1. \tag{74}$$

We start with the case in which electron collisions with neutrals are dominant everywhere in the discharge.

$$\overline{D_{\perp eN}}\,\overline{N}(0) \gg \overline{D_{\perp ei}} \Longrightarrow \frac{\overline{D_{\perp eN}}\exp\left[-\left(\overline{D_{\perp ei}}/2\right)\right]}{\left[1 + \overline{D_{\perp eN}}\sqrt{\pi/\left(2\overline{D_{\perp ei}}\right)}\,\mathrm{erf}\left(\sqrt{\overline{D_{\perp ei}}/2}\right)\right]} \gg \overline{D_{\perp ei}}. \tag{75}$$

This condition can be satisfied only if

$$\overline{D_{\perp ei}} \ll 1. \tag{76}$$

Then

$$\overline{N} \cong \frac{1}{1 + \overline{D_{\perp eN}}\,\overline{n}}, \qquad d \cong \overline{D_{\perp eN}}. \tag{77}$$

Condition (75) is then satisfied if

$$\overline{D_{\perp ei}} \ll 1, \overline{D_{\perp eN}}. \tag{78}$$

In the second case electron collisions with ions are dominant. This cannot hold everywhere as near the boundary the plasma density is zero. We nevertheless neglect the diffusion induced by collisions with neutrals everywhere when

$$\overline{D_{\perp eN}} \ll \overline{D_{\perp ei}}. \tag{79}$$

For simplicity we restrict the analysis of this case to low neutral depletion. If $\overline{D_{\perp ei}} \ll 1$, then (77) holds here too, the neutral depletion is again $d \cong \overline{D_{\perp eN}}$ and condition (79) assures us that neutral depletion is low. This second case thus holds when

$$\overline{D_{\perp eN}} \ll \overline{D_{\perp ei}} \ll 1. \tag{80}$$

5.2. Relation between neutral and plasma densities—nonambipolar diffusion

We turn to the nonambipolar flow at the 'short-circuit' limit. We employ (59) and (60) to obtain

$$\frac{\partial \overline{n}}{\partial \overline{N}} = -\frac{\overline{n}/\overline{D_{\perp ie}} + \overline{N}/\overline{D_{\perp iN}}}{\overline{N}}, \tag{81}$$

where

$$\overline{D_{\perp ie}} \equiv \frac{n_0 D_{\perp ie}}{N_W D_N} = \frac{T_i k_{iN}}{T_g k_{ie}}, \qquad \overline{D_{\perp iN}} \equiv \frac{n_0 D_{\perp iN}}{N_W D_N} = \frac{n_0 T_i}{N_W T_g}. \tag{82}$$

The solution is

$$\overline{n} = \frac{\overline{N}/\overline{D_{\perp iN}}}{1 + 1/\overline{D_{\perp ie}}}\left[\left(\frac{1}{\overline{N}}\right)^{1+1/\overline{D_{\perp ie}}} - 1\right], \tag{83}$$

so that

$$1 = \frac{\overline{N}(0)/\overline{D_{\perp iN}}}{1 + 1/\overline{D_{\perp ie}}}\left[\left(\frac{1}{\overline{N}(0)}\right)^{1+1/\overline{D_{\perp ie}}} - 1\right]. \tag{84}$$

Explicit expressions for the rate of neutral depletion can be derived at appropriate limits. Let us first assume that ion collisions with neutrals are more frequent than ion collisions with electrons everywhere in the discharge. Thus, we require that

$$\overline{D_{\perp ie}} \gg \frac{\overline{D_{\perp iN}}}{\overline{N}(0)}, 1. \tag{85}$$

When these relations hold, there is a pressure balance between ions and neutrals:

$$\overline{D_{\perp iN}}\,\overline{n} + \overline{N} = 1 \Longrightarrow nT_i + NT_g = N_W T_g. \tag{86}$$

At the other limit we assume that ion collisions with electrons are dominant

$$\overline{D_{\perp ie}} \ll \overline{D_{\perp iN}}, \tag{87}$$

an inequality that cannot be valid near the boundary where the plasma density vanishes. We also restrict our analysis to the low depletion case so that $1 - \overline{N}(0) \ll 1$. It is easy to show that if we assume that $(1/\overline{N}(0))^{1+1/\overline{D_{\perp ie}}}$ is about unity, condition (87) cannot be satisfied. We assume that

$$\left(\frac{1}{\overline{N}(0)}\right)^{1+1/\overline{D_{\perp ie}}} \gg 1, \tag{88}$$

so that, because of the low depletion

$$\overline{D_{\perp ie}} \ll 1. \tag{89}$$

We then obtain

$$\left(\frac{\overline{D_{\perp iN}}}{\overline{D_{\perp ie}}}\right)^{\overline{D_{\perp ie}}} \cong \frac{1}{\overline{N}(0)} = d + 1. \tag{90}$$

Since we assume that $d \ll 1$, we write

$$d \cong \overline{D_{\perp \mathrm{ie}}} \ln\left(\frac{\overline{D_{\perp \mathrm{iN}}}}{\overline{D_{\perp \mathrm{ie}}}}\right) \ll 1 \Longrightarrow \overline{D_{\perp \mathrm{iN}}} \ll \overline{D_{\perp \mathrm{ie}}} \exp\left(\frac{1}{\overline{D_{\perp \mathrm{ie}}}}\right). \tag{91}$$

The conditions for ion–electron collisions to be dominant while neutral depletion is low are therefore

$$\overline{D_{\perp \mathrm{ie}}} \ll \overline{D_{\perp \mathrm{iN}}} \ll \overline{D_{\perp \mathrm{ie}}} \exp\left(\frac{1}{\overline{D_{\perp \mathrm{ie}}}}\right). \tag{92}$$

In section 6 we examine the plasma balance equation and the plasma density profile for the nonlinear cross-field ambipolar diffusion. In section 6.1 we analyze the case in which transport is induced by electron collisions with neutrals according to (78) and in section 6.2 the case in which the transport is induced by electron collisions with ions, according to (80). In section 7 we examine the plasma balance equation and the plasma density profile for the nonlinear cross-field nonambipolar diffusion. In section 7.1 we analyze the case in which transport is impeded by ion collisions with neutrals according to (85) and in section 7.2 the case in which the transport is impeded by ion collisions with electrons, according to (92).

6. Nonlinear cross-field ambipolar diffusion

For simplicity the analysis here will be performed in a slab geometry where the variation is along the x coordinate and across the magnetic field. The resulting nonlinear diffusion equation is

$$\frac{\partial}{\partial \xi}\left(D_{\perp \mathrm{e}} \frac{\partial \overline{n}}{\partial \xi}\right) = -\beta N_{\mathrm{W}} a^2 \overline{N}\,\overline{n}, \qquad \xi \equiv \frac{x}{a}, \tag{93}$$

where $\overline{N}$ is given in (71).

6.1. Diffusion induced by electron collisions with neutrals

First, let us assume that (78) holds, so that $\overline{N}$ is described by (77) and the electron mobility is induced by collisions with neutrals. At this limit the ratio of the plasma flux at the boundary to the maximal plasma density becomes

$$cf_{\mathrm{s}} = -\frac{D_{\perp \mathrm{eN}}}{a} \frac{\partial \overline{n}}{\partial \xi}(1). \tag{94}$$

The diffusion equation is then

$$\frac{\partial}{\partial \xi}\left(\overline{N} \frac{\partial \overline{n}}{\partial \xi}\right) = \alpha_{\perp \mathrm{eN}} \overline{N}\,\overline{n}, \qquad \alpha_{\perp \mathrm{eN}} \equiv \frac{\beta N_{\mathrm{W}} a^2}{D_{\perp \mathrm{eN}}} = \frac{m_{\mathrm{e}} \omega_{\mathrm{e}}^2 a^2 \beta}{T_{\mathrm{e}} k_{\mathrm{eN}}}. \tag{95}$$

Let us first present the linear diffusion equation obtained for a negligible neutral depletion, which, according to (77), holds when

$$\overline{D_{\perp \mathrm{eN}}} \ll 1 \Longrightarrow N \cong N_{\mathrm{W}}. \tag{96}$$

The equation is first integrated to

$$\left(\frac{\partial \overline{n}}{\partial \xi}\right)^2 + \alpha_{\perp \mathrm{eN}}\left(1 - \overline{n}^2\right) = 0, \tag{97}$$

so that

$$cf_{\mathrm{s}} = -\frac{D_{\perp \mathrm{eN}}}{a} \frac{\partial \overline{n}}{\partial \xi}(1) = \frac{D_{\perp \mathrm{eN}}}{a} \sqrt{\alpha_{\perp \mathrm{eN}}} = \sqrt{\frac{T_{\mathrm{e}} k_{\mathrm{eN}} \beta N_{\mathrm{W}}^2}{m_{\mathrm{e}} \omega_{\mathrm{e}}^2}}, \tag{98}$$

and further to

$$\int_{\overline{n}}^{1} \frac{\mathrm{d}\overline{n}'}{\sqrt{1 - \overline{n}'^2}} = \arccos(\overline{n}) = \sqrt{\alpha_{\perp \mathrm{eN}}}\,\xi. \tag{99}$$

The plasma balance equation (or the solvability condition), obtained by employing the boundary condition $\overline{n}(\xi = 1) = 0$, results in

$$\int_{0}^{1} \frac{\mathrm{d}\overline{n}'}{\sqrt{1 - \overline{n}'^2}} = \arccos(0) = \frac{\pi}{2} = \sqrt{\alpha_{\perp \mathrm{eN}}}, \qquad \overline{n} = \cos\left(\frac{\pi}{2}\xi\right). \tag{100}$$

The ratio of plasma flux to the maximal density thus turns out, in this linear case, to be

$$cf_{\mathrm{s}} = \frac{D_{\perp \mathrm{eN}}}{a} \frac{\pi}{2} = \frac{2}{\pi} \beta N_{\mathrm{W}} a. \tag{101}$$

Note that the value of the neutral density N_{W} does not enter into the expression for $\alpha_{\perp \mathrm{eN}}$, so that the temperature is not affected by the neutral density. However, a larger neutral density increases the plasma cross-field mobility and, as a result, the ratio of plasma flux to density. In this slab geometry a cosine replaces the Bessel function of the cylindrical geometry and the eigenvalue differs from that in cylindrical geometry by a numerical factor.

From equation (101) we can express the maximal plasma density as a function of the plasma radial flux density

$$n_0 = \frac{m_{\mathrm{e}} \omega_{\mathrm{e}}^2 a}{T_{\mathrm{e}} k_{\mathrm{eN}} N_{\mathrm{W}}} \Gamma_{\mathrm{ir}}(a). \tag{102}$$

The dependence is linear.

We now turn to the nonlinear diffusion that arises due to neutral depletion. Note that $d \cong \overline{D_{\perp \mathrm{eN}}}$. The dimensionless equation

$$\frac{\partial}{\partial \xi}\left[\frac{1}{(1 + d\overline{n})} \frac{\partial \overline{n}}{\partial \xi}\right] = -\alpha_{\perp \mathrm{eN}} \frac{\overline{n}}{(1 + d\overline{n})} \tag{103}$$

is first integrated to

$$\frac{1}{(1 + d\overline{n})} \frac{\partial \overline{n}}{\partial \xi} = -\frac{\sqrt{2\alpha_{\perp \mathrm{eN}}}}{d} \sqrt{\frac{1}{1+d} - \frac{1}{1 + d\overline{n}} + \ln\left(\frac{1+d}{1 + d\overline{n}}\right)} \tag{104}$$

from which the ratio of plasma flux to maximal density is found to be

$$cf_{\mathrm{s}} = \frac{D_{\perp \mathrm{eN}}}{a} \frac{\sqrt{2\alpha_{\perp \mathrm{eN}}}}{d} \sqrt{\frac{1}{1+d} - 1 + \ln(1+d)} = \frac{\beta N_{\mathrm{W}} a}{\sqrt{\alpha_{\perp \mathrm{eN}}}} \frac{\sqrt{2}}{d} \sqrt{\frac{1}{1+d} - 1 + \ln(1+d)}. \tag{105}$$

Equation (104) is further integrated to

$$\int_{\ln(1+d\bar{n})}^{s_0} \frac{\mathrm{d}s}{\sqrt{\exp(-s_0)-\exp(-s)-s+s_0}} = \sqrt{2\alpha_{\perp eN}}\xi,$$

$$s_0 \equiv \ln(1+d) = \ln\frac{N_W}{N(0)}.$$

From the boundary condition $\bar{n} = 0$ we obtain the plasma balance equation (or the solvability condition) as

$$\int_0^{s_0} \frac{\mathrm{d}s}{\sqrt{\exp(-s_0)-\exp(-s)-s+s_0}} = \sqrt{2\alpha_{\perp eN}}, \qquad (106)$$

which relates the rate of neutral depletion d to $\alpha_{\perp eN}$. The density profile is a function of the parameter d alone when we write

$$\frac{\int_{\ln(1+d\bar{n})}^{s_0}\mathrm{d}s[\exp(-s_0)-\exp(-s)-s+s_0]^{-1/2}}{\int_0^{s_0}\mathrm{d}s[\exp(-s_0)-\exp(-s)-s+s_0]^{-1/2}} = \xi, \qquad (107)$$

$$s_0 = \ln(1+d).$$

It is easy to show that when $d \ll 1$ the linear case, (99)–(101), is recovered. In the opposite case of a large neutral depletion, $d \gg 1$, we approximate for $d\bar{n} \gg 1$,

$$\bar{n} = \left(\frac{1}{d}+1\right)\exp\left(-\frac{\alpha_{\perp eN}\xi^2}{2}\right) - \frac{1}{d}, \qquad d \gg 1, \qquad (108)$$

so that the relation between d and $\alpha_{\perp eN}$ becomes

$$\alpha_{\perp eN} = 2\ln(1+d) = 2\ln\frac{N_W}{N(0)}, \qquad d \gg 1. \qquad (109)$$

The approximated form of the ratio of plasma flux to the maximal plasma density becomes

$$cf_s = \frac{D_{\perp eN}}{a}\frac{2\ln(1+d)}{d} = \frac{\beta N_W a}{d}, \qquad d \gg 1. \qquad (110)$$

We can express the plasma density profile as

$$\bar{n} = \left(\frac{1}{d}+1\right)(1+d)^{-\xi^2} - \frac{1}{d}$$
$$= \frac{1}{1/\bar{N}(0)-1}\left[\left(\frac{1}{\bar{N}(0)}\right)^{1-\xi^2} - 1\right], \qquad d \gg 1. \qquad (111)$$

The above approximate relations cease to be valid near the plasma edge where $d\bar{n} \leqslant 1$.

These expressions can be further approximated as

$$\alpha_{\perp eN} = 2\ln d, \qquad \bar{n} = \exp\left[-(\ln d)\xi^2\right],$$
$$cf_s = 2\frac{D_{\perp eN}}{a}\frac{\ln d}{d}, \qquad d \gg 1. \qquad (112)$$

The peaking of the density is expressed as

$$\xi\left[\bar{n} = \exp(-1)\right] = \frac{1}{\sqrt{\ln d}}, \qquad d \gg 1, \qquad (113)$$

while when $\bar{n} = \cos[(\pi/2)\xi]$ that value is $\xi[\bar{n} = \exp(-1)] = 0.76$. As neutral depletion increases the plasma flux decreases for a specified maximal density. Alternatively we can say that for the same rate of plasma production the plasma density is higher, expressing an increased confinement. Employing equation (112) we express the maximal plasma density as a function of the plasma radial flux as

$$n_0 = \frac{T_g m_e \omega_e^2}{m_i k_{iN} T_e k_{eN} N_W}\exp\left[\frac{m_i k_{iN} a}{2T_g}\Gamma_{ir}(a)\right]. \qquad (114)$$

The increase of confinement is expressed in the exponential dependence of the plasma density on the plasma flux.

6.2. Diffusion induced by electron collisions with ions

As a second case we assume that (80) holds, so that the electron cross-field transport is determined by collisions with ions and neutral depletion is small. We express the ratio of plasma flux at the boundary to the maximal plasma density as

$$cf_s = -\frac{D_{\perp ei}}{a}\bar{n}(1)\frac{\partial\bar{n}}{\partial\xi}(1). \qquad (115)$$

Equation (93) becomes a nonlinear diffusion equation of the form

$$\frac{\partial}{\partial\xi}\left(\bar{n}\frac{\partial\bar{n}}{\partial\xi}\right) = -\alpha_{\perp ei}\bar{n}, \qquad \alpha_{\perp ei} \equiv \frac{\beta N_W a^2}{D_{\perp ei}} = \frac{m_e\omega_e^2 a^2\beta}{T_e k_{ei}}\frac{N_W}{n_0}. \qquad (116)$$

The equation is first integrated to

$$\bar{n}\frac{\partial\bar{n}}{\partial\xi} = -\sqrt{\frac{2\alpha_{\perp ei}}{3}}\sqrt{1-\bar{n}^3} \qquad (117)$$

from which we find the ratio of plasma flux to maximal density to be

$$cf_s = D_{\perp ei}\sqrt{\frac{2\alpha_{\perp ei}}{3}} = \frac{2}{3}\frac{\beta N_W a}{\sqrt{\alpha_{\perp ei}}}. \qquad (118)$$

Integrating equation (117) further yields

$$\int_{\bar{n}}^1 \frac{\bar{n}'\mathrm{d}\bar{n}'}{\sqrt{1-\bar{n}'^3}} = \sqrt{\frac{2\alpha_{\perp ei}}{3}}\xi.$$

The plasma balance equation obtained from the boundary condition $\bar{n}(1) = 0$ is

$$\int_0^1 \frac{\bar{n}\mathrm{d}\bar{n}}{\sqrt{1-\bar{n}^3}} = 0.862\,37 = \sqrt{\frac{2\alpha_{\perp ei}}{3}}. \qquad (119)$$

The nonlinear solvability condition includes, as expected, the plasma maximal density n_0. The plasma density profile is described implicitly by

$$\int_{\bar{n}}^1 \frac{\bar{n}'\mathrm{d}\bar{n}'}{\sqrt{1-\bar{n}'^3}} = 0.862\,37\xi. \qquad (120)$$

The transport induced by electrons makes the plasma density flatter than in the linear case. Thus

$$cf_s = 0.862\,37\frac{D_{\perp ei}}{a} = \frac{\beta N_W a}{1.293\,6}. \qquad (121)$$

The ratio of plasma flux to density is larger when electrons collide with ions (for the same $\beta N_W a$). The dependence of the maximal plasma density on the plasma flux in this case is

$$n_0 = \sqrt{\frac{m_e\omega_e^2 a}{0.862\,37\times T_e k_{ei}}\Gamma_{ir}(a)}. \qquad (122)$$

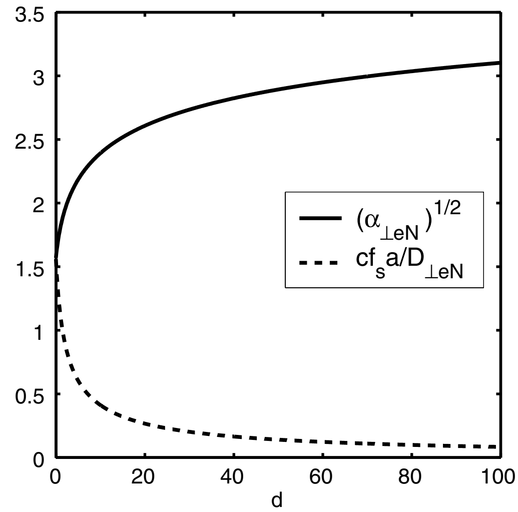

Figure 1. Ambipolar nonlinear cross-field diffusion induced by electron collisions with neutrals. Shown are $\sqrt{\alpha_{\perp\mathrm{eN}}}$ and $cf_{\mathrm{s}}a/D_{\perp\mathrm{eN}}$ as a function of the rate of neutral depletion d, according to (105) and (106). When $d = 0$, no neutral depletion, $\sqrt{\alpha_{\perp\mathrm{eN}}} = cf_{\mathrm{s}}a/D_{\perp\mathrm{eN}} = \pi/2$. When $d \gg 1$ these dependencies are well described by the approximations in (112).

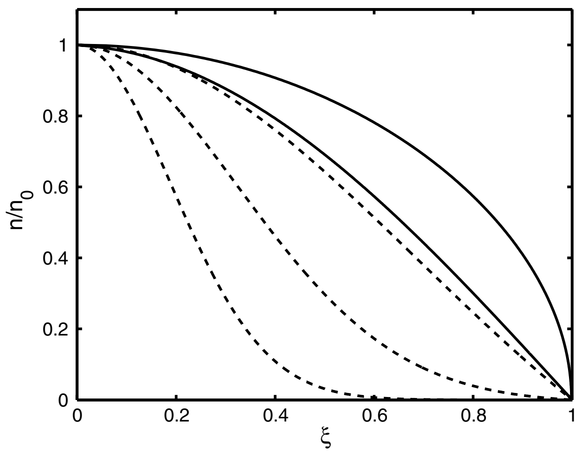

Figure 2. Ambipolar nonlinear cross-field diffusion - the plasma density profiles. The upper solid line shows the plasma radial density profile when diffusion is induced by electron collisions with ions (120). The lower lines show the plasma radial density profiles when the diffusion is induced by electron collisions with neutrals for different rates of the neutral depletion: $d = 0$ (the lower solid line) and $d = 1$, 100, 10^6 (the lower dashed lines). The profiles are calculated according to (107) and for $d = 100$ and $d = 10^6$ are well described by the approximations in (112).

6.3. Numerical examples

Figure 1 shows $\sqrt{\alpha_{\perp\mathrm{eN}}}$ and $cf_{\mathrm{s}}a/D_{\perp\mathrm{eN}}$ as functions of the rate of neutral depletion d, calculated through (106) and (105) as described in section 6.1 for the case that the diffusion is induced by electron collisions with neutrals. When $d = 0$, the linear case with no neutral depletion, the values of these variables are $\sqrt{\alpha_{\perp\mathrm{eN}}} = cf_{\mathrm{s}}a/D_{\perp\mathrm{eN}} = \pi/2$ [(100) and (101)]. For $d \gg 1$ these dependences are well described by the approximations in (112).

Figure 2 shows the plasma radial density profiles for various values of d, calculated through (107) as described

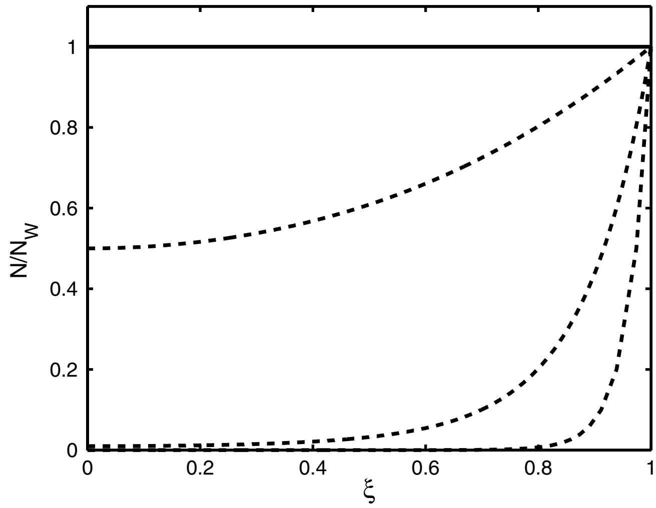

Figure 3. Ambipolar nonlinear cross-field diffusion induced by electron collisions with neutrals. Shown are the neutral density profiles N/N_{W} for different rates of the neutral depletion: $d = 0$ (the solid line) and $d = 1$, 100, 10^6 (the lower dashed lines), calculated according to (77) and (107).

in section 6.1 for the case that the diffusion is induced by electron collisions with neutrals. For $d = 100$ and even more for $d = 10^6$ these density profiles are well described by the approximations in (112). Also shown is the plasma radial density profile for the case in section 6.2 in which the diffusion is induced by electron collisions with ions (120). Figure 3 shows the neutral density profiles N/N_{W} for various values of d, calculated through (107) and (77), as described in section 6.1 for the case that the diffusion is induced by electron collisions with neutrals.

In order to determine the regimes of validity of the two different analyses, we write the electron–ion [54], the electron–neutral and ion–neutral collision rate constants ([17, p 63]) as

$$k_{\mathrm{ei}} = 2.91 \times 10^{-6} \ln\Lambda\, T_{\mathrm{e}}^{-3/2}\ \mathrm{cm^3\,s^{-1}},$$
$$k_{\mathrm{eN}} = 3.85 \times 10^{-8} \alpha_{\mathrm{R}}^{1/2}\ \mathrm{cm^3\,s^{-1}}, \qquad (123)$$
$$k_{\mathrm{iN}} = 8.99 \times 10^{-10} \left(\frac{\alpha_{\mathrm{R}}}{A_{\mathrm{R}}}\right)^{1/2}\ \mathrm{cm^3\,s^{-1}},$$

where A_{R} is the reduced mass in atomic mass units and α_{R} is the relative polarizability. In the expression for k_{ei} the electron temperature T_{e} should be specified in electron-volts. For both regimes it is required that

$$\overline{D_{\perp\mathrm{ei}}} = \frac{T_{\mathrm{e}} m_{\mathrm{i}} k_{\mathrm{iN}} k_{\mathrm{ei}} n_0^2}{T_{\mathrm{g}} m_{\mathrm{e}} \omega_{\mathrm{e}}^2}$$
$$= 1.55 \times 10^{-33} \left(\frac{\alpha_{\mathrm{R}}}{A_{\mathrm{R}}}\right)^{1/2} \frac{A n_0^2}{T_{\mathrm{g}} B^2 T_{\mathrm{e}}^{1/2}} \ll 1. \qquad (124)$$

Here A is the atomic mass number, B is in Tesla, the temperatures are in electron-volts and n_0 is in $\mathrm{cm^{-3}}$. In our case in which ions and neutrals are of the same mass $A_{\mathrm{R}} = A/2$.

The electron dynamics is governed by electron–neutral collisions when (78) is satisfied. That happens when

$$\frac{k_{\mathrm{ei}} n_0}{k_{\mathrm{eN}} N_{\mathrm{W}}} = 75.6 \ln\Lambda \frac{n_0}{\alpha_{\mathrm{R}}^{1/2} T_{\mathrm{e}}^{3/2} N_{\mathrm{W}}} \ll 1. \qquad (125)$$

The results shown in figure 1 as well as the results described by the lower solid line and by the dashed lines in figure 2 and of the results described by the dashed lines in figure 3 are valid when this last inequality holds. We employ the expression for the ionization rate constant, when the electron distribution is Maxwellian,

$$\beta = \sigma_0 v_{\rm te}\left(1+\frac{2T_{\rm e}}{\epsilon_{\rm i}}\right)\exp\left(-\frac{\epsilon_{\rm i}}{T_{\rm e}}\right)$$
$$= 4.4\times 10^{-6}\frac{T_{\rm e}^{1/2}}{\epsilon_{\rm i}{}^2}\left(1+\frac{2T_{\rm e}}{\epsilon_{\rm i}}\right)\exp\left(-\frac{\epsilon_{\rm i}}{T_{\rm e}}\right)\ {\rm cm^3\,s^{-1}}, \quad (126)$$

where $v_{\rm te}\equiv(8T_{\rm e}/\pi m_{\rm e})^{1/2}$ is the electron thermal velocity and $\sigma_0\equiv\pi\left(e^2/4\pi\epsilon_0\epsilon_{\rm i}\right)^2$, ϵ_0 being the permittivity of the vacuum and $\epsilon_{\rm i}$ the ionization energy, $T_{\rm e}$ and $\epsilon_{\rm i}$ expressed in electron-volts. Therefore,

$$\alpha_{\perp\rm eN} = \frac{m_{\rm e}\omega_{\rm e}^2 a^2\beta}{T_{\rm e}k_{\rm eN}}$$
$$= 2\times 10^{13}\frac{B^2a^2}{\alpha_{\rm R}^{1/2}\epsilon_{\rm i}^2 T_{\rm e}^{1/2}}\left(1+\frac{2T_{\rm e}}{\epsilon_{\rm i}}\right)\exp\left(-\frac{\epsilon_{\rm i}}{T_{\rm e}}\right), \quad (127)$$

where B is in Tesla and a in meter. As an example let us take an argon discharge, so that $A=40$, $\epsilon_{\rm i}=15.6\,{\rm eV}$ and $\alpha_{\rm R}=11.1$. Let us assume that $B=0.01\,{\rm T}$, $a=0.05\,{\rm m}$, $T_{\rm e}=5\,{\rm eV}$, $T_{\rm g}=0.027\,{\rm eV}$ and $N_{\rm W}=2\times10^{15}\,{\rm cm^{-3}}$. Then $\alpha_{\perp\rm eN}\cong 5$ and $d=\overline{D_{\perp\rm eN}}\cong 5$. This means that $n_0=10^{13}\,{\rm cm^{-3}}$ and $\overline{D_{\perp\rm ei}}\cong 0.2$.

An example for electron–ion collisions that are dominant is with the same parameters except that the neutral density is lower, $N_{\rm W}=7\times10^{13}\,{\rm cm^{-3}}$. Then

$$\alpha_{\perp\rm ei}=\alpha_{\perp\rm eN}\frac{k_{\rm eN}}{k_{\rm ei}}\frac{N_{\rm W}}{n_0}\cong 1.12, \qquad \overline{D_{\perp\rm ei}}\cong 0.2,$$
$$\overline{D_{\perp\rm eN}}\cong 0.035. \quad (128)$$

7. Nonlinear cross-field nonambipolar diffusion

At the 'short-circuit' limit, when (59) and (60) are used, the cross-field nonambipolar diffusion equation is written as

$$\frac{\partial}{\partial\xi}\left(D_{\perp\rm i}\frac{\partial\overline{n}}{\partial\xi}\right) = -\beta N_{\rm W}a^2\overline{N}\overline{n}, \qquad \xi\equiv\frac{x}{a}. \quad (129)$$

The neutral density and the plasma density are related through (83).

7.1. Diffusion impeded by ion collisions with neutrals

We first analyze the case in which (85) holds, so that ion diffusion is impeded by collisions with neutrals. The ratio of plasma flux to the maximal plasma density is

$$cf_{\rm s} = -\frac{D_{\perp\rm iN}}{a}\frac{\partial\overline{n}}{\partial\xi}(1), \quad (130)$$

and the diffusion equation becomes

$$\frac{\partial}{\partial\xi}\left(\frac{1}{\overline{N}}\frac{\partial\overline{n}}{\partial\xi}\right) = -\alpha_{\perp\rm iN}\overline{N}\overline{n},$$
$$\alpha_{\perp\rm iN}\equiv\frac{\beta N_{\rm W}a^2}{D_{\perp\rm iN}} = \frac{m_{\rm i}k_{\rm iN}\beta N_{\rm W}^2a^2}{T_{\rm i}}. \quad (131)$$

There is a pressure balance between ions and neutrals, $nT_{\rm i}+NT_{\rm g}=N_{\rm W}T_{\rm g}$. The resulting form of the diffusion equation and the following analysis are equivalent to those in [40] and [50], where in this case ion pressure replaces the electron pressure.

The diffusion equation takes the form

$$\frac{\partial}{\partial\xi}\left[\frac{1}{(1-\sin\theta_0\overline{n})}\frac{\partial\overline{n}}{\partial\xi}\right] = -\alpha_{\perp\rm iN}(1-\sin\theta_0\overline{n})\,\overline{n},$$
$$\sin\theta_0\equiv\overline{D_{\perp\rm iN}} = \frac{n_0T_{\rm i}}{N_{\rm W}T_{\rm g}}. \quad (132)$$

Integrating this equation once, we obtain

$$\left[\frac{1}{(1-\sin\theta_0\overline{n})}\frac{\partial\overline{n}}{\partial\xi}\right]^2 = \alpha_{\perp\rm iN}\left(1-\overline{n}^2\right). \quad (133)$$

From this form we find that

$$cf_{\rm s} = \frac{D_{\perp\rm iN}}{a}\sqrt{\alpha_{\perp\rm iN}}. \quad (134)$$

Further defining

$$\overline{n} = \cos\theta, \quad (135)$$

we integrate the equation to

$$\alpha_{\perp\rm iN}^{1/2}\xi = \int_0^\theta\frac{{\rm d}\theta'}{(1-\sin\theta_0\cos\theta')}$$
$$= \frac{1}{\cos\theta_0}{\rm arccot}\left(\frac{\cos\theta-\sin\theta_0}{\sin\theta\cos\theta_0}\right). \quad (136)$$

The arccot function takes values in the interval $[0,\pi]$. With the boundary condition $\xi(\theta=\pi/2)=1$ we obtain the relation

$$\alpha_{\perp\rm iN}^{1/2}\cos\theta_0 = \theta_0+\frac{\pi}{2}. \quad (137)$$

This last equation is in the form of Kepler's equation. The density profile can therefore be written as

$$\frac{\cos\theta-\sin\theta_0}{\sin\theta\cos\theta_0} = \cot\left[\left(\theta_0+\frac{\pi}{2}\right)\xi\right]. \quad (138)$$

We can express these results with the rate of neutral depletion. Note that

$$d\equiv\frac{N_{\rm W}}{N(0)}-1 = \frac{\sin\theta_0}{1-\sin\theta_0}, \quad (139)$$

so that (137), for example, can be written as

$$\alpha_{\perp\rm iN}^{1/2}\frac{\sqrt{1+2d}}{1+d} = \arccos\frac{\sqrt{1+2d}}{1+d}+\frac{\pi}{2}. \quad (140)$$

The neutral density is

$$\frac{N}{N_{\rm W}} = 1-\frac{d\overline{n}}{1+d}, \quad (141)$$

and the ratio of plasma flux to the maximal density becomes

$$cf_{\rm s} = \frac{D_{\perp\rm iN}}{a}\frac{\arccos\left[\sqrt{1+2d}/(1+d)\right]+\pi/2}{\sqrt{1+2d}/(1+d)}. \quad (142)$$

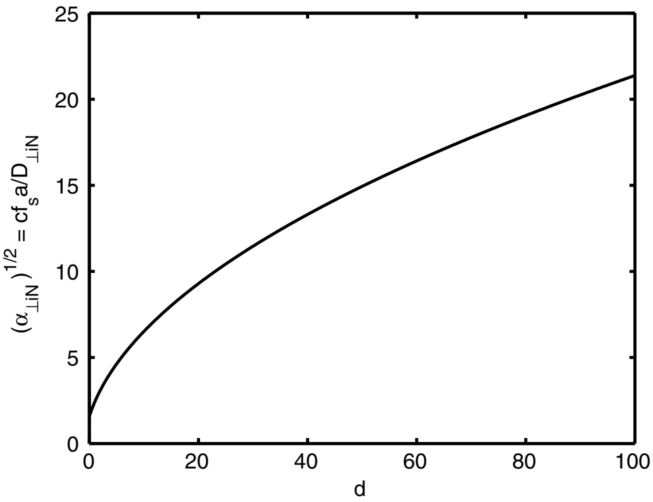

Figure 4. Nonambipolar nonlinear cross-field diffusion impeded by ion collisions with neutrals. Shown are $\alpha_{\perp iN}^{1/2} = cf_s a/D_{\perp iN}$ as a function of the rate of neutral depletion d, according to (140) and (142). $\alpha_{\perp iN}^{1/2} = cf_s a/D_{\perp iN} = \pi/2$ when $d = 0$ and for $d \gg 1$ their values are well approximated by (143).

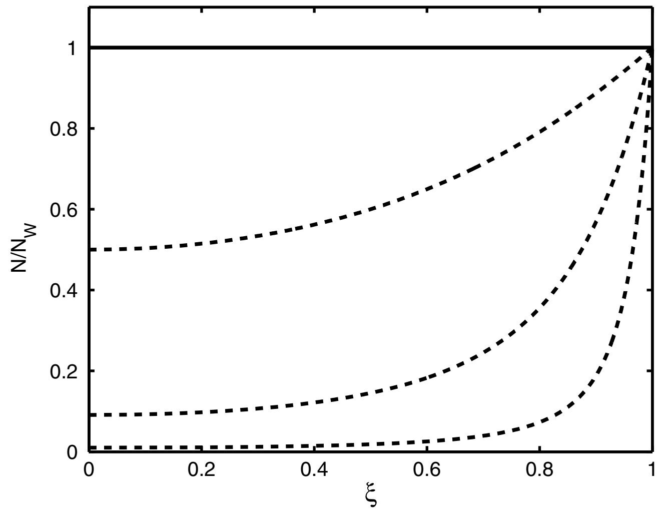

Figure 6. Nonambipolar nonlinear cross-field diffusion impeded by ion collisions with neutrals: the neutral radial density profiles for $d = 0$ (solid line) and for $d = 1, 10, 100$ (dashed lines), according to (141).

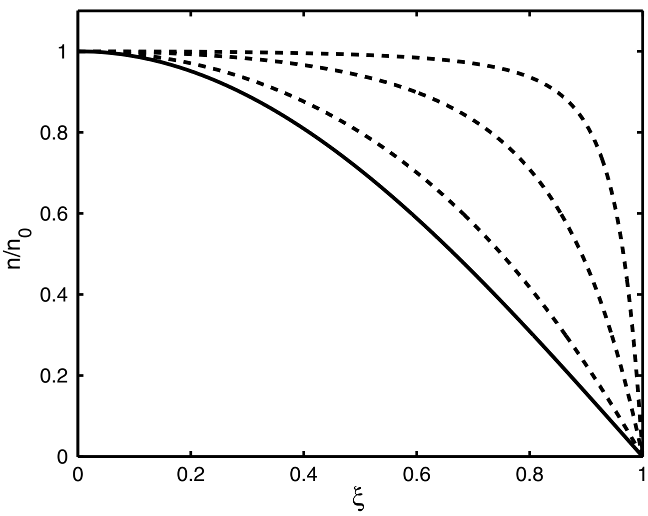

Figure 5. Nonambipolar nonlinear cross-field diffusion impeded by ion collisions with neutrals: the plasma radial density profiles for $d = 0$ (solid line) and for $d = 1, 10, 100$ (dashed lines), according to (138). With the increase of neutral depletion the plasma density becomes more flattened.

At the limit of a high neutral depletion we obtain

$$d \gg 1 \Longrightarrow \alpha_{\perp iN}^{1/2} = \pi\sqrt{\frac{d}{2}} \Longrightarrow cf_s = \pi\frac{D_{\perp iN}}{a}\sqrt{\frac{d}{2}}. \qquad (143)$$

Figure 4 shows $\sqrt{\alpha_{\perp iN}} = cf_s a/D_{\perp iN}$ as a function of the rate of neutral depletion d for this case in which the diffusion is impeded by ion collisions with neutrals. These dependencies are given by (140) and by (142). When $d = 0$ they take the value $\alpha_{\perp iN}^{1/2} = cf_s a/D_{\perp iN} = \pi/2$ while for $d \gg 1$ their values are well approximated by (143).

Figure 5 shows the plasma radial density profiles for various values of d for this case in which the diffusion is impeded by ion collisions with neutrals (138). With the increase in neutral depletion the plasma density becomes more flattened. Figure 6 shows the neutral density profiles N/N_W for various values of d for this case (141). The results shown in figures 5 and 6 are similar to those shown in [40] and in [42], although here it is the ion pressure, rather than the electron pressure, that balances the neutral pressure.

The dependence of the maximal plasma density n_0 on the plasma flux $\Gamma_{ir}(a)$ when $d = 0$ is linear $n_0 = (2m_i k_{iN} N_W a/\pi T_i)\Gamma_{ir}(a)$. When $d \gg 1$ we approximate $d \cong 1/(1 - \sin\theta_0)$ and solve for n_0 to obtain

$$n_0 = \frac{N_W T_g}{T_i}\left[1 - \left(\frac{\pi T_g}{m_i k_{iN} a}\right)^2 \frac{1}{2\Gamma_{ir}(a)^2}\right]. \qquad (144)$$

The density does not exceed a value that is determined by pressure balance between the ions and the neutral gas.

7.2. Diffusion impeded by ion collisions with electrons

We turn to the case that (92) holds, characterized by ion motion being mostly impeded by collisions with electrons and a low neutral depletion. The importance of this case has been pointed out in [5], where it was analyzed for the more applicable cylindrical geometry. The analysis here in a slab geometry allows us to obtain analytical results. The ratio of plasma flux to the maximal plasma density is

$$cf_s = -\frac{D_{\perp ie}}{a\bar{n}(1)}\frac{\partial \bar{n}}{\partial \xi}(1), \qquad (145)$$

and the governing equation becomes

$$\frac{\partial}{\partial \xi}\left(\frac{1}{\bar{n}}\frac{\partial \bar{n}}{\partial \xi}\right) = -\alpha_{\perp ie}\bar{n},$$
$$\alpha_{\perp ie} \equiv \frac{\beta N_W a^2}{D_{\perp ie}} = \frac{m_i k_{ie}\beta N_W n_0 a^2}{T_i}. \qquad (146)$$

The quantity $\alpha_{\perp ie}$ is identical to q in [5]. Integrating this equation once we obtain

$$\left(\frac{1}{\overline{n}}\frac{\partial \overline{n}}{\partial \xi}\right)^2 = \alpha_{\perp ie}\left(1-\overline{n}\right). \quad (147)$$

From here we get that

$$cf_s = \frac{D_{\perp ie}}{a}\sqrt{\alpha_{\perp ie}}. \quad (148)$$

We integrate the equation further to

$$\overline{n} = 1 - \left[\frac{1-\exp\left(-\sqrt{\alpha_{\perp ie}}\xi\right)}{1+\exp\left(-\sqrt{\alpha_{\perp ie}}\xi\right)}\right]^2. \quad (149)$$

The density vanishes at infinity only. We modify the boundary condition to be

$$v_i\left(\xi=1\right) = \frac{1}{4}\sqrt{\frac{8T_i}{\pi m_i}}. \quad (150)$$

The ion velocity is found to be

$$v_i\left(\xi=1\right) = -\frac{D_{\perp ie}}{a\overline{n}^2\left(1\right)}\frac{\partial \overline{n}}{\partial \xi}\left(1\right) = \frac{D_{\perp ie}\sqrt{\alpha_{\perp ie}}}{a\overline{n}\left(1\right)}. \quad (151)$$

From the last two equations we find that

$$\overline{n}\left(\xi=1\right) = \frac{D_{\perp ie}}{a}\sqrt{\frac{2\pi m_i \alpha_{\perp ie}}{T_i}}. \quad (152)$$

The plasma balance equation becomes

$$\frac{D_{\perp ie}}{a}\sqrt{\alpha_{\perp ie}}\sqrt{\frac{2\pi m_i}{T_i}} = 1 - \left[\frac{1-\exp\left(-\sqrt{\alpha_{\perp ie}}\right)}{1+\exp\left(-\sqrt{\alpha_{\perp ie}}\right)}\right]^2. \quad (153)$$

Combining the plasma balance equation with (148) we obtain

$$\overline{n}(\xi=1) = cf_s\sqrt{\frac{2\pi m_i}{T_i}} = 1 - \left[\frac{1-\exp(-\sqrt{\alpha_{\perp ie}})}{1+\exp(-\sqrt{\alpha_{\perp ie}})}\right]^2. \quad (154)$$

For the diffusion approximation to be valid $\overline{n}(1)$ should be much smaller than unity. The plasma balance equation is then approximated as

$$\sqrt{\alpha_{\perp ie}} \gg 1 \Longrightarrow \sqrt{\alpha_{\perp ie}}\exp\left(\sqrt{\alpha_{\perp ie}}\right) = \sqrt{\frac{8T_i}{\pi m_i}}\frac{a}{D_{\perp ie}}. \quad (155)$$

Figure 7 shows the plasma radial density profiles for various values of $\sqrt{\alpha_{\perp ie}}$ according to (149) for the case analyzed here in which the ion diffusion is impeded by collisions with electrons (149). The plasma density becomes peaked with the increase in plasma density, similarly to what was first shown in [5] for a cylindrical geometry. For a higher plasma density the ions collide more with electrons and their diffusion slows.

In addition to the peaking of the density profile, the slowing cross-field diffusion due to a larger $\sqrt{\alpha_{\perp ie}}$ also results in a decrease in cf_s, the ratio of plasma flux to the maximal

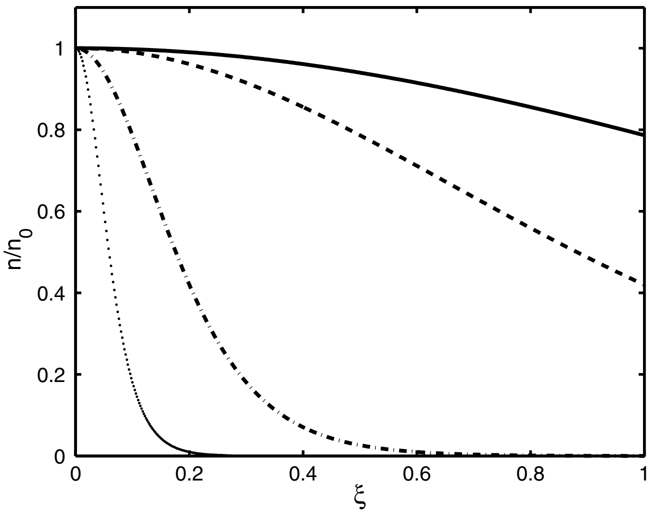

Figure 7. Nonambipolar nonlinear cross-field diffusion impeded by ion collisions with electrons: the plasma radial density profile for various values of $\sqrt{\alpha_{\perp ie}}$ according to (149). The plasma density becomes peaked with the increase of plasma density, as was shown in [5] for a cylindrical geometry.

plasma density. This decrease in cf_s is reflected in figure 7 in the decrease in $\overline{n}(\xi=1) = cf_s\sqrt{2\pi m_i/T_i}$ as $\sqrt{\alpha_{\perp ie}}$ increases, also as expected from (154) (for a specified ion temperarure).

We derive further an approximate relation between the maximal plasma density and the plasma flux. When equation (155) holds we approximate $\sqrt{\alpha_{\perp ie}} \cong \ln[(\sqrt{8T_i/\pi m_i}a/D_{\perp ie})/\ln(\sqrt{8T_i/\pi m_i}a/D_{\perp ie})]$. In this case we approximate equation (154) as $cf_s = \sqrt{8T_i/\pi m_i}\exp(-\sqrt{\alpha_{\perp ie}})$. The plasma flux is thus $\Gamma_{ir}(a) = n_0\sqrt{8T_i/\pi m_i}\ln(\sqrt{(8T_i/\pi m_i)}a/D_{\perp ie})/(\sqrt{8T_i/\pi m_i}a/D_{\perp ie})$ and therefore the maximal plasma density is

$$n_0 = \left(\frac{T_i}{m_i k_{ie} a}\right)\sqrt{\frac{\pi m_i}{8T_i}}\exp\left[\frac{m_i k_{ie} a}{T_i}\Gamma_{ir}\left(a\right)\right]. \quad (156)$$

In this case the maximal plasma density depends exponentially on the plasma flux, exhibiting an enhanced confinement.

7.3. Numerical examples

The regime of validity of the analysis in section 7.1 is defined by

$$\overline{D_{\perp ie}} = \frac{T_i k_{iN}}{T_g k_{ie}} = \frac{8.99\times 10^{-10}\sqrt{\alpha_R/A_R}AT_e^{1/2}\left(3/2\right)T_i^2}{1.6\times 10^{-9}\ln\Lambda T_g} \gg \frac{n_0 T_i}{\overline{N}\left(0\right)N_W T_g}, \qquad 1, \quad (157)$$

where we used the expression for the ion–electron collision rate constant with Maxwellian ions ([54] as in [5]):

$$k_{ie} = \frac{1.6\times 10^{-9}\ln\Lambda}{AT_e^{1/2}(3/2)T_i}\,\mathrm{cm^3\,s^{-1}}. \quad (158)$$

For an argon discharge in which $T_e = 3$ eV, $T_i = 0.3$ eV and $T_g = 0.027$ eV, we obtain $\overline{D_{\perp ie}} = T_i k_{iN}/T_g k_{ie} \cong 24.3$. If

$a = 0.05$ m, and $N_W = 10^{21}$ m^{-3}, we find that

$$\alpha_{\perp iN} = 4.1 \times 10^{-35} \left(\frac{\alpha_R T_e}{A_R}\right)^{1/2} \times \frac{(1 + 2T_e/\epsilon_i)\exp(-\epsilon_i/T_e)}{T_i \epsilon_i{}^2} (N_W a)^2 \cong 27.7. \qquad (159)$$

The neutral depletion is

$$d \cong \frac{2\alpha_{\perp iN}}{\pi^2} \cong 5.6 \Longrightarrow \frac{1}{\overline{N}(0)} \cong 6.6. \qquad (160)$$

The plasma density turns out to be $n_0 = N_W(T_g/T_i)d/(1+d) = 7.6 \times 10^{19}$ m^{-3}.
For this case of large neutral depletion, inequality (157) becomes

$$\overline{D_{\perp ie}} >> d,$$

which is reasonably satisfied here.

The regime of validity of the analysis in section 7.2 is defined as

$$1 \ll \frac{k_{ie} n_0}{k_{iN} N_W} \ll \exp\left(\frac{T_g k_{ie}}{T_i k_{iN}}\right). \qquad (161)$$

For an argon discharge in which $T_e = 3$ eV, $T_i = T_g = 0.027$ eV, we obtain $T_i k_{iN}/T_g k_{ie} \cong 0.12$. If n_0 and N_W are of a similar value the analysis here is valid. We find that

$$\alpha_{\perp ie} = \frac{4.9 \times 10^{-22}}{T_i^2 \epsilon_i{}^2} \left(1 + \frac{2T_e}{\epsilon_i}\right) \exp\left(-\frac{\epsilon_i}{T_e}\right) N_W n_0 a^2, \qquad (162)$$

where the densities should be specified in cm^{-3} and a in meters. Equation (155) is satisfied for $N_W = n_0 = 10^{12}$ cm^{-3} and $a = 0.05$ m, and $\alpha_{\perp ie}$ is found to be 4.2.

8. Summary

In this paper we analyzed the steady state of a cylindrical plasma diffusing both axially along a confining magnetic field and radially across the field. We first assumed that the transport coefficients are constant. This assumption and a variable separation, even without the assumption of ambipolarity, allowed us to write the equations in a linear form and to derive an analytical 2D description of the plasma and of the electric potential. We used the derived analytical solutions to examine two special linear cases. In one case our general solution is reduced to the ambipolar-flow solution, while in the second case the nonambipolar flow exhibits the 'short circuit' effect.

In the second part of the paper we allowed the transport coefficients to depend on the plasma or neutral density, which made the equations nonlinear. We investigated the nonlinear 1D cross-field transport, first when the flow is ambipolar and then at the 'short-circuit' limit of the nonambipolar flow. For each flow we examined two cases, one in which the transport is governed by collisions with neutrals and a second in which the transport is governed by collisions with plasma particles.

The ambipolar cross-field diffusion is governed by the dynamics of the electrons which are magnetized. Electron collisions enhance the cross-field diffusion. If the electron collisions are with ions, an increase in the plasma (and ion) density increases the number of electron collisions and, as a result, was shown to enhance the cross-field diffusion and to flatten the plasma density profile. If, however, the electron collisions are with neutrals, an increase in the plasma density that is followed by a neutral depletion decreases the number of electron collisions and, as a result, was shown to reduce the cross-field diffusion and to make the plasma density more peaked at the center of the discharge.

At the 'short-circuit' limit of the nonambipolar flow the cross-field diffusion is determined by the ions that were assumed in this paper to be unmagnetized. Collisions in the case of unmagnetized diffusing ions impede the diffusion. The role of collisions in affecting the rate of diffusion and the plasma density profile at this 'short-circuit' limit is therefore opposite to their role in the ambipolar case in which the diffusing electrons were assumed magnetized. Thus, if ion collisions are with the electrons, an increase in the plasma (and electron) density increases the number of ion collisions and, as a result, was shown to reduce the cross-field diffusion and to make the plasma density profile more peaked at the center of the discharge, as was previously shown in [5]. If, however, ion collisions are with neutrals, an increase in the plasma density that is followed by a neutral depletion decreases the number of ion collisions and, as result, was shown to enhance the cross-field diffusion and to make the plasma density more flattened.

The analysis here exhibited the rich and complex plasma nonlinear cross-field transport. We identified the dimensionless parameters that characterize the discharge. We will use this analysis in a subsequent paper to examine how the discharge characteristics vary as functions of the varied neutral-gas pressure, magnetic field intensity and deposited wave-power. We note that we described only some of the possible governing processes. One important case that we did not address here is that of neutrals that move ballistically and are coupled to the plasma through ionization only. Another important case is of a significant neutral depletion in a plasma in which cross-field diffusion is dominated by electron–ion collisions. These cases will be examined in future studies.

The analysis presented here is aimed mostly at understanding crossed-field transport in low temperature plasmas. The analysis should also be relevant to other plasma configurations in which cross-field transport is dominant. One such example is the transport barrier in tokamaks that arises where the transport coefficient varies rapidly [55].

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