Modification of short scale-length tearing modes by the Hall field

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The calculation of the growth rate of tearing modes is extended to short scale lengths by including the Hall field. A unified dispersion relation is found that describes usual tearing modes at one limit and the Hall tearing modes with the enhanced growth rate at the opposite limit. The dispersion relation is valid for both collisional and collisionless plasmas.

I. INTRODUCTION

Recently there have been extensive studies of processes that occur on scale lengths between the electron and the ion skin depths and on the time scale between the electron and the ion cyclotron periods. The limit of immobile ions and mobile electrons only is called electron magnetohydrodynamics (EMHD).[1] The magnetic field evolution is governed by the Hall field. Within this approximation the magnetic field was shown to penetrate the plasma due to nonuniformities,[2–4] or as a whistler wave.[5,6] Such magnetic field penetration occurs in plasma opening switches (POS) and in plasma beams. The dominance of electron dynamics was demonstrated for short time scales in space plasmas[7] and in laser-produced plasmas.[8–10]

In this paper we examine the modification of tearing modes by the Hall field for short scale lengths. This modification was first studied by Gordeev,[11] who neglected the ion dynamics, and assumed hot collisional electrons. Hassam studied the modification in the collisional regime, allowing ion motion.[12] Recently, Seyler[13] used two fluid models to study plasma stability in the lower-hybrid frequency range. The plasma was assumed cold and collisionless. This stability problem was studied recently also by Bulanov *et al.*[14]

Our approach is similar to Seyler's. However, instead of a numerical solution of the full nonlinear equations, we study analytically the linear stability problem. We derive a unified relation that describes usual tearing modes at one limit and the EMHD modes at the opposite limit. The plasma pressure is neglected. The dispersion relation is valid for both collisional and collisionless plasmas.

In Sec. II we derive the general dispersion relation. The collisional case is examined in Sec. III and the collisionless case in Sec. IV. In both cases the instability is enhanced by the Hall field.

II. DERIVATION OF THE DISPERSION RELATION

The governing equations are the continuity equation,

$$\frac{\partial n}{\partial t}+\nabla\cdot n\mathbf{u}=0, \tag{1}$$

the momentum equation,

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}=\frac{1}{Mnc}\mathbf{J}\times\mathbf{B}, \tag{2}$$

the generalized Ohm's law,

$$\mathbf{E}+\frac{\mathbf{u}\times B}{c}=\frac{\mathbf{J}\times\mathbf{B}}{enc}+\frac{m}{e}\left(\frac{\partial}{\partial t}-\frac{\mathbf{J}}{en}\cdot\nabla\right)\left(\frac{\mathbf{J}}{en}\right)+\eta\mathbf{J}, \tag{3}$$

Ampère's law,

$$\frac{4\pi}{c}\mathbf{J}=\nabla\times\mathbf{B}, \tag{4}$$

and Faraday's law,

$$-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}=\nabla\times\mathbf{E}. \tag{5}$$

In these equations n is the plasma density (quasineutrality is assumed), M and $\mathbf{u}$ are the ion mass and flow velocity, $\mathbf{J}$ is the current, η is the resistivity, $\mathbf{E}$ and $\mathbf{B}$ are the electric and the magnetic fields, m is the electron mass, e is the elementary charge, and c is the velocity of light in vacuum. We assumed that the electron and ion pressures are negligible in Ohm's law and in the momentum equation. We also neglected the displacement current in Ampère's law and the derivatives of the ion velocities in Ohm's law. These equations with zero resistivity were also the basis for Seyler's analysis.[13]

We are interested in the stability of a current layer parallel to an external magnetic field. We could allow small but finite parallel wave numbers and use a reduced form of the equations similar to reduced magnetohydrodynamics (MHD).[15] For simplicity, however, the parallel wave number is assumed zero, $\partial/\partial z=0$. We write the dimensionless magnetic field $\mathbf{b}$ and flow velocity $\mathbf{u}$ as

$$\begin{aligned}\mathbf{b}&=\hat{z}\times\nabla\Psi+\hat{z}(1+\delta b),\\ \mathbf{u}&=\hat{z}\times\nabla\phi+\nabla\chi+v\hat{z}.\end{aligned} \tag{6}$$

The magnetic field $\mathbf{b}$ is normalized to the external magnetic field B_0, and $\mathbf{u}$ and v are normalized to $u_A\equiv c\omega_{ci}/\omega_{pi}$, where $\omega_{ci}\equiv eB_0/Mc$ and $\omega_{pi}^2\equiv 4\pi n_0e^2/M$, n_0 is the assumed uniform equilibrium density. The length is normalized to c/ω_{pi}. This set of equations is a simplified version of Hall MHD.[16,17,7] We examine the stability of a current layer for which the magnetic field is of the form

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$$\mathbf{b}=\hat{z}\times\hat{x}\frac{\partial\Psi_0(x)}{\partial x}+\hat{z}, \tag{7}$$

where $\partial\Psi_0/\partial x=-\epsilon F(x)$ and $\epsilon\ll 1$. We write the equations for the linearized quantities,

$$\Psi=\Psi_0(x)+\Psi_1(x)\exp(\gamma\tau+iky),$$
$$\delta b=b_1(x)\exp(\gamma\tau+iky),$$
$$n=1+n_1, \tag{8}$$
$$\chi=\chi_1(x)\exp(\gamma\tau+iky),$$
$$\phi=\phi_1(x)\exp(\gamma\tau+iky).$$

We use dimensionless time $\tau(\equiv\omega_{ci}t)$ and density $n(\equiv n/n_0)$. The governing equations (1)–(5) become

$$[\gamma-(\gamma\delta^2+\nu)\nabla^2]\Psi=\epsilon F\frac{\partial\chi}{\partial x}-ik\epsilon F\phi+ik\epsilon Fb, \tag{9}$$

$$[\gamma-(\gamma\delta^2+\nu)\nabla^2]b=-\nabla^2\chi+ik\epsilon(-F''+F\nabla^2)\Psi+ik\epsilon^2FF'n, \tag{10}$$

$$\gamma\nabla^2\phi=-ik\epsilon(-F''+F\nabla^2)\Psi, \tag{11}$$

$$\gamma\chi=-b, \tag{12}$$

$$\gamma n+\nabla^2\chi=0, \tag{13}$$

where the normalized resistivity is $\nu\equiv\eta n_0ec/B_0$, and $\delta^2\equiv m/M$.

If χ and b are small these equations take the form of the standard tearing mode problem.[18] If, however, χ and ϕ are small, the equations become the EMHD equations for tearing modes that include electron dynamics only.[14] We would like to examine the general solution of these equations and to examine the transition between the two opposite limits.

Using Eqs. (12) and (13) we can eliminate n and χ. Equations (9)–(11) become

$$[\gamma-(\gamma\delta^2+\nu)\nabla^2]\Psi=-\epsilon F\left(\frac{1}{\gamma}\frac{\partial b}{\partial x}+ik(\phi+b)\right), \tag{14}$$

$$\left[\gamma-\left(\gamma\delta^2+\nu+\frac{1}{\gamma}-\frac{ik\epsilon^2FF'}{\gamma^2}\right)\nabla^2\right]b=ik\epsilon(-F''+F\nabla^2)\Psi, \tag{15}$$

$$\gamma\nabla^2\phi=-ik\epsilon(-F''+F\nabla^2)\Psi. \tag{16}$$

The first term on the right-hand side (rhs) of Eq. (14) and the last term on the left-hand side (lhs) of Eq. (15) result from the change of density due to the magnetic pressure. These two terms change the structure of the equations in the layer. For simplicity in the present study we restrict ourselves to cases in which we may neglect these terms. We therefore require that

$$\left|\frac{1}{\gamma}\frac{\partial b}{\partial x}\right|\ll k|\phi+b|, \tag{17}$$

and that

$$\left|\gamma\delta^2+\nu+\frac{1}{\gamma}\right|\gg\left|\frac{k\epsilon^2FF'}{\gamma^2}\right|. \tag{18}$$

Thus, the governing equations become

$$[\gamma-(\gamma\delta^2+\nu)\nabla^2]\Psi=-i\epsilon Fk(\phi+b), \tag{19}$$

$$\left[\gamma-\left(\gamma\delta^2+\nu+\frac{1}{\gamma}\right)\nabla^2\right]b=ik\epsilon(-F''+F\nabla^2)\Psi, \tag{20}$$

$$\gamma\nabla^2\phi=-ik\epsilon(-F''+F\nabla^2)\Psi. \tag{21}$$

These equations have a standard tearing mode form. The exterior solution satisfies the equations

$$(-F''+F\nabla^2)\Psi=0, \tag{22}$$

$$\gamma\Psi=-ik\epsilon F(\phi+b). \tag{23}$$

We assume that

$$F=\tanh(x/l), \tag{24}$$

and therefore

$$\Psi(x)=\exp(\pm kx)[1\pm\tanh(x/l)/kl]. \tag{25}$$

Following (18) and (24) we require that

$$\gamma^2\delta^2+\nu\gamma+1\ll k\epsilon^2/\gamma l. \tag{26}$$

We also define the jump, in the derivative,

$$\Delta'\equiv\Psi'(0^+)-\Psi'(0^-)=2[(1/kl^2)-k]. \tag{27}$$

The equations inside the tearing layer are approximated as

$$\gamma\Psi(0)-(\gamma\delta^2+\nu)\Psi''=-ik\epsilon(x/l)(\phi+b), \tag{28}$$

$$-(\gamma\delta^2+\nu+1/\gamma)b''=ik\epsilon(x/l)\Psi'', \tag{29}$$

$$\gamma\phi''=-ik\epsilon(x/l)\Psi''. \tag{30}$$

These equations are combined to

$$\frac{d^2g}{d\zeta^2}-\zeta^2g=\zeta, \tag{31}$$

where

$$g\equiv\left(\frac{il(\gamma\delta^2+\nu)}{k\epsilon}\right)\times\left[\frac{(\nu+1/\gamma+\gamma)}{(\gamma\delta^2+\nu+1/\gamma)(\gamma\delta^2+\nu)\gamma}\left(\frac{k\epsilon}{l}\right)^2\right]^{3/4}\phi \tag{32}$$

and

$$\zeta\equiv x/x_t. \tag{33}$$

The width of the tearing layer is

$$x_t=\left[\frac{(\gamma\delta^2+\nu+1/\gamma)(\gamma\delta^2+\nu)\gamma}{(\nu+1/\gamma+\gamma)}\left(\frac{l}{k\epsilon}\right)^2\right]^{1/4}. \tag{34}$$

The condition (17) now becomes

$$|b|\ll k\gamma x_t|\phi+b|. \tag{35}$$

Matching the external solution with the internal solution, we obtain the general dispersion relation:

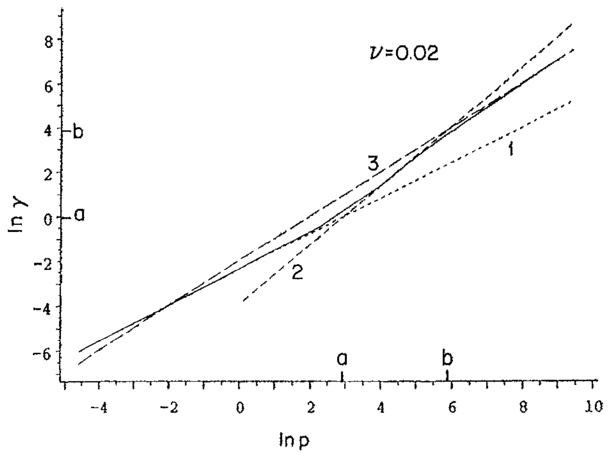

FIG. 1. The growth rate γ versus the parameter p (both on the logarithmic scale) in the collisional case. Curve 1 shows the MHD growth rate [Eq. (43)], curve 2 the transition regime [Eq. (46)] and curve 3 the EMHD regime [Eq. (49)]. On the $\ln(p)$ axis a denotes where $p=1/\nu^{3/4}$ and b denotes $p=1/\nu^{3/2}$. The solid curve is the solution of the dispersion relation (41). On the $\ln \gamma$ axis a denotes where $\gamma=1$ and b where $\gamma=1/\nu$. In the figure $\nu=0.02$.

$$\gamma\left(\frac{(\gamma\delta^2+\nu+1/\gamma)\gamma}{(\nu+1/\gamma+\gamma)(\gamma\delta^2+\nu)^3}\right)^{1/4}=p, \tag{36}$$

where

$$p\equiv\frac{\Delta'}{I}\left(\frac{k\epsilon}{l}\right)^{1/2}; \qquad I\equiv\pi\Gamma(\tfrac{3}{4})/\Gamma(\tfrac{1}{4}). \tag{37}$$

Since ν and δ^2 are much smaller than unity, we may now write (35) as

$$\left(\gamma+\frac{1}{\gamma}\right)(\gamma\delta^2+\nu)\frac{k}{\gamma}\frac{\Delta'}{I}\gg 1. \tag{38}$$

In the following we discuss separately the collisional and the collisionless cases.

III. THE COLLISIONAL CASE

The collisional case is characterized by

$$\nu\gg\gamma\delta^2. \tag{39}$$

We require that the growth rate will be larger than the rate of diffusion,

$$\gamma l^2\gg\nu. \tag{40}$$

Following (39), the dispersion relation (36) becomes

$$\gamma\left(\frac{(\nu\gamma+1)}{(\nu+1/\gamma+\gamma)\nu^3}\right)^{1/4}=p. \tag{41}$$

Figure 1 shows the growth rate γ as a function of p. For

$$p\ll 1/\nu^{3/4}, \tag{42}$$

the growth rate is

$$\gamma=p^{4/5}\nu^{3/5}. \tag{43}$$

This is the standard tearing mode problem. From (42) and (43) we obtain that

$$\gamma\ll 1. \tag{44}$$

The growth time is longer than the ion cyclotron period. This is the regime that $\phi\gg b$.

In the intermediate regime,

$$1/\nu^{3/4}\ll p\ll 1/\nu^{3/2}, \tag{45}$$

the growth rate is

$$\gamma=p^{4/3}\nu. \tag{46}$$

In this regime,

$$1\ll\gamma\ll 1/\nu. \tag{47}$$

This is an intermediate regime. The Hall field and ion dynamics are both important. The field b is determined by the ion compression.

The third domain is the EMHD domain. If

$$p\gg 1/\nu^{3/2}, \tag{48}$$

the growth rate becomes[11,12,14]

$$\gamma=p\nu^{1/2}. \tag{49}$$

In this regime,

$$\gamma\gg 1/\nu. \tag{50}$$

The solid curve in Fig. 1 shows the solution of Eq. (41). The dotted curves denoted 1, 2, and 3 are the curves of (43), (46), and (49), respectively. To the left of a (where $p=1/\nu^{3/4}$) the solid line coincides with curve 1. Between a and b (where $p=1/\nu^{3/2}$), the solid line coincides with curve 2. To the right of b, the solid line coincides with curve 3.

In the collisional case, condition (38) becomes

$$\frac{k\Delta'}{I}\gg p^{8/5}\nu^{1/5}, \tag{51a}$$

for $\gamma\ll 1$, and

$$\frac{k\Delta'}{I}\gg\frac{1}{\nu}, \tag{51b}$$

for $\gamma\gg 1$.

For all three domains to be possible (39) has to be satisfied,

$$p\ll\nu^{1/2}/\delta^2, \tag{52}$$

or for the domain (48) to exist,

$$\frac{1}{\nu^{3/2}}\ll\frac{\nu^{1/2}}{\delta^2} \tag{53}$$

or

$$\nu\gg\delta. \tag{54}$$

The collision frequency has to be larger than the lower-hybrid cyclotron frequency. This is a common condition for the existence of resistive EMHD.[19]

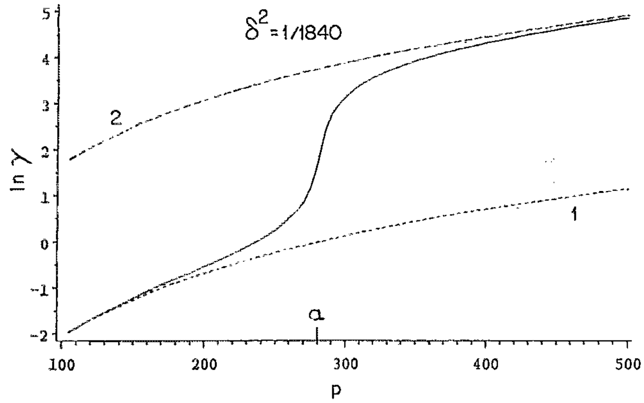

FIG. 2. The growth rate γ (on the logarithmic scale) versus the parameter p in the collisionless case. Curve 1 shows the MHD growth rate [Eq. (59)], while curve 2 shows the EMHD growth rate [Eq. (61)]. The solid curve shows the curve (57). On the p axis a denotes $p=1/\delta^{3/2}$. In the figure $\delta^2=\frac{1}{1840}$.

IV. THE COLLISIONLESS CASE

We turn now to the collisionless case and assume that

$$\nu \ll \gamma\delta^2. \tag{55}$$

The dispersion relation becomes a second-order polynomial for γ^2,

$$\frac{(\gamma^2\delta^2+1)\gamma^2}{(\gamma^2+1)\delta^6}=p^4. \tag{56}$$

The solution for the growth rate is

$$\gamma^2=\left(\frac{p^4\delta^6-1}{2\delta^2}\right)+\left[\left(\frac{p^4\delta^6-1}{2\delta^2}\right)^2+p^4\delta^4\right]^{1/2}. \tag{57}$$

Figure 2 shows γ vs p. For

$$p \ll 1/\delta^{3/2}, \tag{58}$$

the growth rate is approximately

$$\gamma=p^2\delta^3. \tag{59}$$

For

$$p \gg 1/\delta^{3/2}, \tag{60}$$

the growth rate becomes

$$\gamma=p^2\delta^2. \tag{61}$$

The expression (61) represents the increase in the growth rate in the EMHD regime relative to the usual MHD regime (59).

Curves 1 and 2 in Fig. 2 show (59) and (61), respectively. The solid curve shows (57). To the left of a (where $p=1/\delta^{3/2}$) the solid line coincides with curve 1, while to the right of a it coincides with curve 2.

In the collisionless case condition (38) becomes

$$\frac{k\Delta'}{I} \gg p^2\delta, \tag{62a}$$

for $\gamma \ll 1$, and

$$\frac{k\Delta'}{I} \gg \frac{1}{(p^2\delta^4)}, \tag{62b}$$

for $\gamma \gg 1$.

In order to have the full collisionless regime, we require that (55) is satisfied; therefore

$$\nu \ll p^2\delta^5 \ll \delta^2, \tag{63}$$

since $p^2\delta^3 \ll 1$. Combining (54) and (63) we see that the collisional regime is the regime where $\nu \gg \delta$, while the collisionless regime is for $\nu \ll \delta^2$. The collision frequency has to be smaller than the ion cyclotron frequency.

V. CONCLUSIONS

We summarize this paper by presenting the main results in dimensional form. Let us start with the collisional case. We define, as is done commonly,

$$\tau_{\rm A}\equiv\frac{(4\pi Mn)^{1/2}}{kB_y}; \quad \tau_{\rm R}\equiv\frac{4\pi l^2}{c^2\eta}, \tag{64}$$

where $B_y \equiv \epsilon B_0$ and all quantities are in dimensional form. We also define the Hall time,

$$\tau_{\rm H}\equiv\left(\frac{4\pi}{c^2}\right)\left(\frac{2nec}{B_y}\right)\left(\frac{2\pi l}{k}\right). \tag{65}$$

The Hall time is equivalent to the resistive time with the Hall "resistivity" $(B_y/2nec)$, and the length $(2\pi l/k)^{1/2}$. Note that $\tau_{\rm H}<\tau_{\rm A}$ when $l<c/\omega_{pi}$. The growth rate in the usual MHD regime (43) is

$$\gamma=\left(\frac{\Delta_0}{I}\right)^{4/5}\tau_{\rm A}^{-2/5}\tau_{\rm R}^{-3/5}. \tag{66}$$

In the second regime (46), the growth rate is

$$\gamma=\left(\frac{\Delta_0}{I}\right)^{4/3}\tau_{\rm A}^{-2/3}\tau_{\rm R}^{-1}\tau_{ci}^{2/3}, \tag{67}$$

where $\tau_{ci}\equiv(Mc/eB_0)$. In the EMHD regime the growth rate (49) becomes

$$\gamma=\left(\frac{\Delta_0}{I}\right)\tau_{\rm R}^{-1/2}\tau_{\rm H}^{-1/2}. \tag{68}$$

Here $\Delta_0=(1/kl-kl)$.

We turn now to the collisionless case. In the collisionless MHD the growth rate (59), is

$$\gamma=\left(\frac{\Delta_0}{I}\right)^2\tau_{\rm A}^{-1}\left(\frac{c}{l\omega_{pe}}\right)^3, \tag{69}$$

where ω_{pe} is the electron plasma frequency. In the EMHD regime (61), the growth rate is

$$\gamma=\left(\frac{\Delta_0}{I}\right)^2\tau_{\rm H}^{-1}4\pi\left(\frac{c}{l\omega_{pe}}\right)^2. \tag{70}$$

We note that in the EMHD regime, whether collisional (68) or collisionless (70), the growth rate is independent of the ion mass.

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