Deviations from the frozen-in law in the presence of small (but nonzero) resistivity

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The freezing of magnetic field flux into an electron fluid is examined during fast magnetic field evolution that results from the nonlinear skin effect. First, fast magnetic field evolution, which does not involve flux penetration, is shown to follow naturally from the frozen-in law. Second, fast evolution is analyzed in which large deviations from the frozen-in law (large flux penetration) occur in the presence of small (but nonzero) resistivity. A direct relation is shown between the deviation from the frozen-in law and the energy dissipated per electron along its orbit.

The evolution of the magnetic field in plasmas of zero resistivity is constrained by the freezing of the magnetic field flux into the electron fluid. This constraint is expected to restrict the evolution of the magnetic field in short duration plasmas of low resistivity, such as in a plasma opening switch.[1,2] A mechanism for fast magnetic field evolution in low-resistivity plasmas has been explored recently.[3–8] The fast evolution is induced by density gradients or by magnetic field curvature and is called the nonlinear skin effect.[3–5,8] Questions related to this issue still remain. Is the fast evolution associated with a true flux penetration? Could a fast flux penetration occur when the resistivity is small? If indeed there is a fast flux penetration, is it associated with a large energy dissipation? The purpose of this Brief Communication is to present two explicit examples that should help clarify these issues. First, a case in which fast magnetic field evolution results naturally from the frozen-in law is discussed and no flux penetration occurs. Then a second case is discussed, in which large deviations from the frozen-in law and fast flux penetration occur even though the resistivity is small. In this case, the flux penetration is shown to be related directly to the energy dissipated.

The familiar frozen-in law is obtained when Ohm's law, in the form

$$\mathbf{E}=\eta\mathbf{j}-\frac{\mathbf{v}_e\times\mathbf{B}}{c}, \tag{1}$$

is combined with Faraday's law, giving

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{v}_e\times\mathbf{B})-c\eta\nabla\times\mathbf{j}. \tag{2}$$

Here $\mathbf{B}$ is the magnetic field, $\mathbf{j}$ is the current, $\mathbf{v}_e$ is the electron flow velocity, η the resistivity, and c the velocity of light in vacuum. In the limit of $\eta=0$ this equation describes the freezing of the magnetic field into the electron fluid. When $\eta\neq 0$, the magnetic field penetrates the electron fluid.

For simplicity, slab geometry is used and it is assumed that $\partial/\partial z=0$ and $\mathbf{B}=\hat{e}_z B(x,y,t)$. Equation (2) becomes

$$\frac{dB}{dt}=-B\nabla\cdot\mathbf{v}_e-c\eta(\nabla\times\mathbf{j})_z, \tag{3}$$

where $d/dt\equiv\partial/\partial t+\mathbf{v}_e\cdot\nabla$ is the convective derivative. Combining Eq. (3) with the continuity equation yields

$$\frac{d}{dt}\left(\frac{B}{n}\right)=\frac{c^2\eta}{4\pi n}\Delta B. \tag{4}$$

Here n is the electron density and Δ is the Laplacian. In this derivation Ampère's law was used and the displacement current was neglected. In the limit of $\eta=0$, the frozen-in law results in the familiar constancy of B/n along an electron orbit. In the cylindrical case with azimuthal symmetry, the magnetic field is $\mathbf{B}=\hat{e}_\theta B(r,z,t)$, and the quantity B/rn is constant along an electron orbit.[9] It is emphasized that the constancy of B/n along an electron orbit and the constancy of B along a current line do not imply that electrons have to move along equidensity contours in order to satisfy the frozen-in law. If the density changes along the electron orbit, the magnetic field could change in time, so that B/n remains constant along the electron orbit. Therefore, if an electron moves from a low-density region to a high-density region, the magnetic field grows in time.

We examine the freezing of the flux, or the constancy of B/n along an electron orbit, in short duration plasma, in which the ions are assumed to be immobile so that the Hall field is dominant. This assumption applies when the time scales of interest are short compared with the ion response time. As a result, the electron velocity is $\mathbf{v}_e = -\mathbf{j}/en$. In the particular case that

$$n=n(y)=1/\alpha y, \tag{5}$$

the magnetic field evolution is governed by Burgers' equation

$$\frac{\partial B}{\partial t}+\frac{c\alpha B}{4\pi e}\frac{\partial B}{\partial x}=\frac{c^2\eta}{4\pi}\left(\frac{\partial^2 B}{\partial x^2}+\frac{\partial^2 B}{\partial y^2}\right). \tag{6}$$

Here, $-e$ is the electron charge. Two cases of fast magnetic field evolution will now be described. In the first case, B/n remains constant along an electron orbit and the fast magnetic field evolution results naturally from the frozen-in law with $\eta=0$. In the second case, a small resistivity allows a large deviation from the frozen-in law, and B/n changes

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significantly during the fast magnetic field evolution. Magnetic field penetration occurs, therefore, only in the second case.

In the first case,

$$B(x,y,t)=\frac{x}{[1+(c\alpha/4\pi e)t]}, \tag{7}$$

so that the electron velocity $v_y=-j_y/en$ is

$$\frac{dy}{dt}=\frac{c\alpha y}{(4\pi e+c\alpha t)}, \tag{8}$$

where $j_y=-(c/4\pi)(\partial B/\partial x)$. Along the electron orbit,

$$y(t)=y(t=0)[1+(c\alpha/4\pi e)t], \tag{9}$$

and the frozen-in law is satisfied, since

$$B(t)/n(t)=x\alpha y(0)=\text{const}, \tag{10}$$

where electrons move only in the y direction so x is constant. The decrease in time of the magnetic field results naturally from the frozen-in law. The fast magnetic field evolution does not involve flux penetration. The electron moves from a high-density region to a low-density region and at the same time the magnetic field decreases, so that B/n remains constant along the electron orbit. A qualitative discussion of the equivalent effect in cylindrical geometry is given in Ref. 6.

In the second case, there is a large deviation from the frozen-in law (large flux penetration). The shock solution of Eq. (6) is given by

$$B=\frac{B_0}{1+\exp\{(B_0\alpha/2\eta ec)[x-(cB_0\alpha/8\pi e)t]\}}. \tag{11}$$

The electrons move in the y direction only. Their orbit is again found from $v_y=-j_y/en$ and $-(cB_0\alpha/8\pi e)(\partial B/\partial x)=(\partial B/\partial t)$, yielding

$$y(t_2)=y(t_1)\exp\{-(2/B_0)[B(t_2)-B(t_1)]\}, \tag{12}$$

where the subscripts 1,2 denote the times t_1,t_2 at which the quantities are evaluated. It is clear that B/n is not constant along an electron orbit. In fact,

$$\frac{B_2}{n_2}-\frac{B_1}{n_1}=\frac{1}{n_1}\left[B_2\exp\left(\frac{2}{B_0}(B_2-B_1)\right)-B_1\right]. \tag{13}$$

The change in the density that an electron experiences found for $B_1=0$, $B_2=B_0$ is

$$n_1/n_2=y_2/y_1=\exp(-2). \tag{14}$$

In this case of a large deviation from the frozen-in law, the magnetic field does penetrate the electron fluid.

A direct relation between the deviation from the frozen-in law (the flux penetration) and the energy dissipation per electron along its orbit, is found by integrating the right-hand side of Eq. (4) along an electron orbit. The integration across the shock yields

$$\left[\frac{B}{n}\right]=\frac{8\pi}{B_0}\int_{-\infty}^{\infty}\frac{dt}{n}\eta j^2. \tag{15}$$

The right-hand side equals the rate of energy dissipation per electron along its orbit multiplied by $8\pi/B_0$. The square brackets on the left-hand side denote the jump of the quantity in brackets across the shock. A large deviation from the frozen-in law (large flux penetration) is associated with a large energy dissipation. Even though the resistivity is small, the resistance and the associated dissipation are large. The resistance is large because of the narrow shock layer. Rewriting Eq. (15) yields

$$n\int_{-\infty}^{\infty}\frac{dt}{n}\eta j^2=\frac{B_0^2}{8\pi}.$$

In the region behind the shock front, the energy dissipated by Joule heating equals the deposited magnetic field energy. This energy equipartition is analyzed using a fluid model in Ref. 8.

In summary, a fast magnetic field evolution is possible even in plasmas of low resistivity. The frozen-in law, which is valid at the zero resistivity limit, allows certain forms of fast magnetic field evolution with no flux penetration. In a different form of evolution, the shock propagation, large deviations from the frozen-in law and large flux penetration is associated with a large energy dissipation. A direct relationship between the deviation from the frozen-in law and the energy dissipated per electron along its orbit is also presented.

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