Thermomagnetic instability in a magnetized plasma

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The thermomagnetic instability, driven by gradients of the plasma density and temperature, which are parallel to each other and orthogonal to the magnetic field, is studied. The focus of this paper is on a magnetized plasma, in which the electron cyclotron frequency is larger than the electron collision frequency. The nature of the transverse heat conductivity in the magnetized plasma causes the instability to be driven by opposite density and temperature gradients. The growth rate in the magnetized plasma is much smaller than in the unmagnetized plasma. In a very strong magnetic field heat convection is larger than heat conduction, and the instability is further stabilized by the Hall field.

I. INTRODUCTION

The generation of strong magnetic fields in laser-produced plasmas is a subject of major concern. Theoretical studies have explored various generation[1-9] and transport[10,11] mechanisms and experiments have been carried out to measure those fields.[12-15] The magnetic fields could influence significantly the spreading of thermal energy as well as the hydrodynamics of the plasma. One of the mechanisms for the spontaneous generation of magnetic fields is the thermomagnetic instability driven by parallel equilibrium density and temperature gradients.[1,2] A perturbed normal temperature gradient generates a magnetic field through the term $\nabla n \times \nabla T$, where n and T are the electron density and temperature. The magnetic field perturbation enhances the temperature perturbation through the magnetic field dependence of the thermal conductivity. The analyses of this instability assumed that the equilibrium plasma was unmagnetized.[1,2] However, hot plasmas ($T = 10\,\text{keV}$), even of density $10^{20}\,\text{cm}^{-3}$, become magnetized ($\Omega\tau > 1$, where Ω is the electron cyclotron frequency and τ is the electron collision time) already for a magnetic field of 300 G.

In this paper we examine the nature of the thermomagnetic instability in a magnetized plasma. We study the linear stability of a plasma of parallel density and temperature gradients immersed in a uniform magnetic field orthogonal to the gradients. In a magnetized plasma ($\Omega\tau \gg 1$) the form of the transverse thermal conductivity is different from its form in the unmagnetized plasma. The conditions for instability as well as the growth rate are shown to be different. The instability in the unmagnetized case occurs when the density and the temperature gradients are in the same direction. We show that in the magnetized case the two gradients have to be in opposite directions. In strongly magnetized plasmas ($\Omega\tau \gg 1$ and also $\beta \equiv 8\pi nT/B^2 \ll 1$, where B is the intensity of the magnetic field), the nature of the instability is different again. The perturbed magnetic field enhances the temperature perturbation not through heat conduction, which is negligible, but through convection. The Hall field in Ohm's law stabilizes the instability that exists only for large ratios of the temperature and density gradients. A related major role of the Hall field in fast convection of magnetic and thermal energies in low-density plasmas has been recently explored.[16,17]

The reduction of the growth rate of the instability when $\Omega\tau > 1$ may imply that this thermomagnetic instability is not the source of the megagauss fields observed in laser-produced plasmas. The existence of the instability in moderately low β plasmas suggests that the instability may arise in tokamaks where large temperature gradients exist.

In Sec. II we present the model, derive the governing linearized equations, and study the roots of the dispersion relation for various values of the characteristic parameters. The results are discussed in Sec. III.

II. THE MODEL

We assume that the process is so fast that the ions are immobile. The time scale is between the electron and the ion cyclotron periods and the spatial scale is between the electron and the ion skin depths. With the further assumption of quasineutrality, the electron density is fixed in time. The equations that govern the evolution of the electron thermal energy and of the magnetic field are the electron heat-balance equation

$$\frac{3}{2}\frac{\partial nT}{\partial t} + \nabla\cdot\left(\frac{3}{2}nT\mathbf{v}\right) + nT\,\nabla\cdot\mathbf{v} + \nabla\cdot\mathbf{q} + \boldsymbol{\pi}\,\frac{\partial \mathbf{v}}{\partial \mathbf{x}} = Q, \tag{1}$$

and Faraday's law

$$-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \nabla\times\mathbf{E}. \tag{2}$$

The other governing equations are the generalized Ohm's law

$$\mathbf{E} = \frac{\mathbf{R}}{en} - \frac{\mathbf{v}\times\mathbf{B}}{c} - \frac{\nabla(nT)}{en} - \frac{\nabla\cdot\boldsymbol{\pi}}{en}, \tag{3}$$

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Ampère's law

$$(4\pi/c)\mathbf{j}=\nabla\times\mathbf{B}, \tag{4}$$

and the relation

$$\mathbf{j}=-en\mathbf{v}. \tag{5}$$

In these equations n, $\mathbf{v}$, T, and π are the electron density, flow velocity, temperature, and stress tensor, $\mathbf{q}$ and Q are the heat flux density and the generated heat, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, $\mathbf{R}$ is the collisional force, $\mathbf{j}$ is the current, $-e$ is the electron charge, and c is the velocity of light in vacuum.

We are interested in the evolution of the thermomagnetic instability in a magnetized plasma in which $\Omega\tau\gg 1$. We therefore neglect all the dissipative terms in the equations. Effects such as the Nernst effect, which are of a dissipative origin, do not appear in our model. We assume, however, that the plasma is collisional enough so that we can neglect the stress tensor. We later derive a condition to justify the neglect of the stress tensor. The heat flux density is approximated as

$$\mathbf{q}=-K_\wedge \hat{h}\times\nabla T, \tag{6}$$

where $\hat{h}\equiv\mathbf{B}/|B|$ and $K_\wedge=K_\wedge(B/n,T)$ is the transverse thermal conductivity. With the neglect of $\mathbf{R}$, Q, and π the governing equations combine to

$$\frac{3}{2}n\frac{\partial T}{\partial t}+\frac{c}{4\pi e}\nabla\times\mathbf{B}\cdot\left(-\frac{3}{2}\nabla T+\frac{T}{n}\nabla n\right)$$
$$-\hat{h}\times\nabla T\cdot\nabla K_\wedge-K_\wedge\nabla T\cdot\nabla\times\hat{h}=0 \tag{7}$$

and

$$\frac{\partial\mathbf{B}}{\partial t}+\frac{c}{4\pi e}\nabla\times\left(\frac{(\nabla\times\mathbf{B})}{n}\times\mathbf{B}\right)+\frac{c}{en}\nabla n\times\nabla T=0. \tag{8}$$

We now study the stability of an equilibrium in which $\nabla n\times\nabla T=0$, and the magnetic field has a constant direction normal to the gradients. We look at perturbations that propagate in a direction perpendicular in both the equilibrium gradients and the magnetic field. We linearize the quantities as

$$n=n(x),$$
$$T=T_0(x)+T_1(x)\exp[i(ky-\omega t)],$$
$$\mathbf{B}=\hat{e}_z\{B_0+B_1(x)\exp[i(ky-\omega t)]\}. \tag{9}$$

The problem is similar to the problem studied by Tidman and Shanny.[1] However, since they looked at the low $\Omega\tau$ limit, they kept the dissipative terms that we neglect. We, on the other hand, retain the convective terms, such as the Hall field in Faraday's law, that they neglected since they treated the weak magnetic field limit. Both in Tidman and Shanny and in our paper the thermal conductivity [the third term in Eq. (7)] and the electron pressure gradient [the last term in Eq. (8)] play major roles. In order to compare the instability in the large and small $\Omega\tau$ regimes, we keep the general form of $K_\wedge$ for arbitrary $\Omega\tau$. Even though our analysis is strictly consistent only in the large $\Omega\tau$ regime, we recover the essential features of the instability for small $\Omega\tau$, as given by Tidman and Shanny, through the correct form of $K_\wedge$. In this way we follow the changing character of the instability with the changing $\Omega\tau$.

Since $K_\wedge=K_\wedge(B/n,T)$, we express the term associated with the heat conductivity as

$$\hat{h}\times\nabla T\cdot[\nabla B-(B/n)\nabla n](a/n),$$

where $a\equiv\partial K_\wedge/\partial(B/n)(B=B_0,n=n_0)$. The linearized equations (7) and (8) become

$$-\frac{3}{2}n\omega T_1+\frac{kc}{4\pi e}B_1\left(-\frac{3}{2}\frac{\partial T_0}{\partial x}+\frac{T_0}{n_0}\frac{\partial n_0}{\partial x}\right)$$
$$-\frac{\partial T_0}{\partial x}\frac{k}{n}aB_1-kT_1\frac{Ba}{n^2}\frac{\partial n_0}{\partial x}=0, \tag{10}$$

$$-\omega B_1+\frac{kc}{en_0}\frac{\partial n_0}{\partial x}\left(\frac{B_0B_1}{4\pi n_0}+T_1\right)=0. \tag{11}$$

The inclusion of dissipative terms would turn the algebraic equations for T_1 and B_1 into differential equations and also introduce dispersion that would stabilize the short wavelength perturbations.

Expressing the gradients as $\partial T_0/\partial x=T_0/l_T$ and $\partial n_0/\partial x=n_0/l_n$, we obtain the dispersion relation for $f=\omega/kc$,

$$f^2-\frac{f}{l_n}\left(-\frac{2}{3}\frac{B_0a}{n^2c}+\frac{B_0}{4\pi en}\right)+\frac{T_0}{enl_n^2}\left[\frac{1}{4\pi e}\left(\eta-\frac{2}{3}\right)\right.$$
$$\left.+\frac{2}{3}\frac{a}{nc}\eta\right]-\frac{2}{3}\frac{a}{c}\frac{B_0^2}{4\pi el_n^2n^3}=0. \tag{12}$$

Here $\eta\equiv l_n/l_T$. The condition for instability is

$$\frac{B^2}{(4\pi en)^2}-\frac{T}{\pi e^2n}\left(\eta-\frac{2}{3}\right)+\frac{4}{9}\frac{B^2a^2}{n^4c^2}$$
$$+\frac{B^2}{3\pi en^3}\frac{a}{c}-\frac{8aT\eta}{3en^2c}<0. \tag{13}$$

We omitted, for convenience, the subscript zeros. The general expression for the transverse thermal conductivity $K_\wedge$ is[18]

$$K_\wedge=\frac{nT\tau}{m}\frac{\Omega\tau[\gamma_1''(\Omega\tau)^2+\gamma_0'']}{\Delta}, \tag{14}$$

where m is the electron mass, $\Delta\equiv(\Omega\tau)^4+\delta_1(\Omega\tau)^2+\delta_0$, and the coefficients are given by Braginskii.[18] When $\Omega\tau$ is small, $K_\wedge\cong nT\tau\Omega\tau\gamma_0''/m\delta_0$, and

$$a\cong n\frac{\beta}{8\pi}\frac{(\Omega\tau)^2\gamma_0''c}{\delta_0 e}>0.$$

The condition for instability is

$$\eta>\left(\frac{2}{3}+\frac{1}{2\beta}\right)\left[1+\frac{32\pi^2}{3}\frac{\gamma_0''}{\delta_0}\left(\frac{\sigma\lambda_D}{c}\right)^2\right]^{-1}, \tag{15}$$

where σ is the plasma conductivity. The expression in the square brackets can also be written as

$$1+\frac{2}{3}\frac{\gamma_0''}{\delta_0}\left(\frac{\lambda_e\omega_p}{c}\right)^2, \tag{16}$$

where ω_p is the plasma frequency, $\lambda_e\equiv v_e\tau$ is the mean-free path, and v_e is the electron thermal velocity. When the second term is very large, inequality (15) reduces to the Tidman–Shanny instability criterion $\eta > 0$. The growth rate in the case that $\Omega\tau \ll 1$ and $\lambda_e \omega_p / c \gg 1$ is

$$\text{Im}\,\omega = 4\pi k \sigma \lambda_D^2 \left(\frac{2}{3} \frac{\gamma_0''}{l_n l_T \delta_0} \right)^{1/2}. \tag{17}$$

A similar expression has been derived by Tidman and Shanny.[1]

We note that in recent theoretical studies a different form for the transverse thermal conductivity has been derived.[19] these recent studies have different predictions about the instability in the low-collisionality case.

In a magnetized plasma ($\Omega\tau \gg 1$) the transverse thermal conductivity is $K_\wedge \cong \frac{5}{2}(cnT/eB)$, and $a = -\frac{5}{2}(cn^2T/eB^2)$. The dispersion relation becomes

$$f^2 - \sqrt{\frac{2}{\beta}} \frac{\lambda_D}{l_n} \left(1 + \frac{5}{6}\beta\right) f + \frac{\lambda_D^2}{l_n^2} \left[1 + \eta\left(1 - \frac{5}{6}\beta\right)\right] = 0. \tag{18}$$

The condition for instability becomes

$$\beta^2(\tfrac{25}{4} + 15\eta) - \beta(3 + 18\eta) + 9 < 0. \tag{19}$$

The instability exists for $-\frac{5}{12} \leqslant \eta \leqslant (2 - \sqrt{10})/3$, or $\eta > (2 + \sqrt{10})/3$ and $\beta_1 \leqslant \beta \leqslant \beta_2$, where

$$\beta_{1,2} = \frac{6 + 36\eta \pm 12\cdot 3^{1/2}(3\eta^2 - 4\eta - 2)^{1/2}}{(25 + 60\eta)}, \tag{20}$$

and also for $\eta < -5/12$ for $\beta > \beta_2$.

Had we retained the stress tensor, we then would have had to add to the left-hand side of Eq. (11) the term

$$-\frac{\eta_0 k^2 c^2}{12\pi e} \left[\frac{\partial}{\partial x}\left(\frac{1}{n_0}\right)\right]^2,$$

the leading viscous term in the magnetized case. Here $\eta_0 \equiv 0.73 n_0 T_0 \tau$.[18] This term can be neglected relative to the second term in Eq. (11) if the inequality

$$(\omega_p \lambda_e^3 / c l_n)(1/l_n l_T)^{1/2} \ll \Omega\tau, \tag{21}$$

is satisfied. Usually the plasma is collisional enough so that Eq. (21) is satisfied and the stress tensor may be neglected.

Rather than studying the general dispersion relation (18) let us compare the stability of the plasma for large and small values of β. Let us first consider $\beta \gg 1$. Because of the sign change of a the instability exists for negative $\eta, \eta < -5/12$. The growth rate becomes

$$\text{Im}\,\omega = kc\left(-\frac{5}{6} \frac{\lambda_D^2}{l_n l_T} \beta\right)^{1/2}. \tag{22}$$

The modification of the instability by the magnetic field is demonstrated by comparing the expressions (17) and (22). While for a weak magnetic field the growth rate is given by Eq. (17). When the magnetic field is substantial, η has to be negative and the growth rate is given by (22). The ratio of the growth rates in the high β limit is

$$\frac{\text{Im}\,\omega(\Omega\tau \gg 1)}{\text{Im}\,\omega(\Omega\tau \ll 1)} = \frac{5}{4}\left(\frac{\beta\delta_0}{\gamma_0''}\right)^{1/2} \frac{c}{\omega_p \lambda_e}. \tag{23}$$

Therefore the plasma magnetization reduces the growth rate by $c/\omega_p \lambda_e$, which is a very small number in laser-produced plasmas. We conclude from (23) that the thermomagnetic instability is not likely to generate the megagauss fields observed in laser-produced plasma because of this reduction of the growth rate for $\Omega\tau > 1$.

We now turn to the low β limit. The effect of the thermal conductivity is not important in the strongly magnetized plasma. The instability exists nevertheless even in the case where $a = 0$. It is driven by the electron pressure gradient and heat convection rather than by heat conduction. Condition (19) results in the following condition for instability:

$$\eta > 1/2\beta. \tag{24}$$

The Hall field due to the large magnetic field stabilizes the plasma in this regime. The growth rate becomes

$$\text{Im}\,\omega = kc(\lambda_D^2 / l_n l_T)^{1/2} = k v_e (c^2/\omega_p^2 l_n l_T)^{1/2}. \tag{25}$$

As could be expected, when the thermal energy of the plasma is smaller than the magnetic field energy, a large temperature gradient is required for the instability to exist and the growth rate is smaller. In the low β limit the growth rate is smaller by $\beta^{1/2}$ than in the magnetized ($\Omega\tau \gg 1$) high β regime. The growth rate is also smaller than kv_e and therefore Landau damping is small. Expression (25) shows that the instability could be nevertheless substantial in low β plasmas. Consider a tokamak plasma of $\beta = 0.1$. For $\eta > 5$ the growth rate of the instability could be substantial. The instability we describe is the electromagnetic analog of certain electrostatic instabilities in low β plasmas.[20,21]

We estimate the maximal growth rate using the assumption that the short wavelength perturbations are stabilized by the resistivity. To order of magnitude

$$(\text{Im}\,\omega)_{\max} \cong c^2 (k_{\max}^2 / 4\pi\sigma). \tag{26}$$

The maximal growth rates are therefore estimated to be

$$\text{Im}\,\omega(\beta \ll 1) \cong 4\pi \lambda_D^2 \sigma / l_n l_T, \tag{27a}$$

$$\text{Im}\,\omega(\beta \gg 1, \Omega\tau \ll 1) = \frac{128}{3} \frac{\pi^2}{c^2} \frac{\sigma^3 \lambda_D^4}{l_n l_T} \frac{\gamma_0''}{\delta_0}, \tag{27b}$$

$$\text{Im}\,\omega(\beta \gg 1, \Omega\tau \gg 1) = \frac{5}{6} 4\pi \frac{\lambda_D^2 \beta \sigma}{|l_n l_T|}. \tag{27c}$$

The growth rate is largest in the unmagnetized plasma.

III. DISCUSSION

We have studied the thermomagnetic instability in a magnetized plasma. We have shown that both the criterion for instability and the growth rate of the instability are different in unmagnetized ($\Omega\tau \ll 1$) and in magnetized ($\Omega\tau \gg 1$) plasmas. The difference results from the different transverse thermal conductivities in the two regimes. When the plasma is strongly magnetized ($\beta \ll 1$) the thermal conductivity is negligible and the instability is stabilized by the Hall field.

In our model we have made several simplifying assumptions. The magnetic field was taken to be uniform. The analysis was local in order to derive a simple algebraic dispersion relation. It is important to know how the relaxation of these assumptions affects the result, and also what the nonlinear evolution of the instability is.

The main conclusions of this paper are the following: First, the growth rate of the thermomagnetic instability decreases significantly when the plasma becomes magnetized. Second, even though the growth rate is smaller, the instability could still be present in low β plasmas, such as tokamaks.

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[1] D. A. Tidman and R. A. Shanny, Phys. Fluids **17**, 1207 (1974).
[2] L. A. Bol'shov, Yu. A. Dreizin, and A. M. Dykhne, JETP Lett. **19**, 168 (1974).
[3] M. G. Haines, J. Plasma Phys. **12**, 1 (1974).
[4] J. H. Brownell, Comments Plasma Phys. Controlled Fusion **4**, 131 (1979).
[5] A. Z. Dolginov and V. A. Urpin, Sov. Phys. JETP **50**, 912 (1979).
[6] M. G. Haines, Phys. Rev. Lett. **47**, 917 (1981).
[7] T. Yabe, K. Mima, T. Sugiyama, and K. Yoshikawa, Phys. Rev. Lett. **35**, 663 (1975).
[8] P. Amendt, H. U. Rahman, and M. Strauss, Phys. Rev. Lett. **53**, 1266 (1984).
[9] A. Nishiguchi, T. Yabe, and M. G. Haines, Phys. Fluids **28**, 3683 (1985).
[10] M. G. Haines, Plasma Phys. Controlled Fusion **28**, 1705 (1986).
[11] T. J. M. Boyd and D. Cooke, Phys. Fluids **31**, 651 (1988).
[12] J. A. Stamper, K. Papadopoulos, R. N. Sudan, S. O. Dean, and E. A. McLean, Phys. Rev. Lett. **26**, 1012 (1971).
[13] J. A. Stamper and B. H. Ripin, Phys. Rev. Lett. **34**, 138 (1974).
[14] A. Raven and O. Willi, Phys. Rev. Lett. **43**, 278 (1979).
[15] M. D. J. Burgess, B. Luther-Davies, and K. A. Nugent, Phys. Fluids **28**, 2286 (1985).
[16] A. S. Kingsep, K. V. Chukbar, and V. V. Yan'kov, in *Reviews of Plasma Physics*, edited by B. Kadomstev (Consultants Bureau, New York, 1990), Vol. 16, p. 243.
[17] A. Fruchtman, Phys. Fluids B **3**, 1908 (1991).
[18] S. I. Braginskii, in *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1, p. 205.
[19] E. M. Epperlein and M. G. Haines, Phys. Fluids **29**, 1029 (1986).
[20] Y. C. Lee, J. Q. Dong, P. N. Guzdar, and C. S. Liu, Phys. Fluids **30**, 1331 (1987).
[21] W. Horton, B. G. Hong, and W. M. Tang, Phys. Fluids **31**, 2971 (1988).