# The snowplow in plasmas of nonuniform density

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The snowplow motion and the convective skin effect are shown to be the two opposite limits in a unified model of plasma pushing by magnetic pressure. During the snowplow motion the plasma is compressed to a high density in a thin layer and the ion velocity equals the shock velocity. If, on the other hand, the spatial scale of the density gradient is smaller than the ion skin depth, the magnetic field penetrates with a velocity much higher than the ion velocity and the plasma compression is small.

## I. INTRODUCTION

Plasma pushing by magnetic pressure is an important process in many laboratory plasmas, such as shock tubes,[1] pinches,[2] magnetically insulated ion diodes,[3] and plasma opening switches.[4] The simplest model which describes this pushing is the snowplow model.[5] According to this classical model the plasma is pushed by the magnetic pressure to a velocity that is determined by a momentum balance. The pushed plasma is compressed to a high density in a narrow layer. The magnetic field penetrates only into this layer which is of a thickness that is determined by the resistivity (for layers thicker than the electron skin depth).

Recently, a mechanism for magnetic field penetration, the convective skin effect, has been explored.[6–11] This penetration is of much interest, since it is expected to occur for times (between the electron and the ion cyclotron periods) and for lengths (between the electron and the ion skin depths) that are characteristic of plasmas in certain pulsed-power devices. The penetration, induced by density nonuniformities or magnetic field curvature, reduces the amount of energy delivered to the ions. The snowplow effect and the convective skin effect are competing processes.

In this paper we present a unified picture of the two processes. We solve an approximated one-dimensional (1-D) model problem that allows for both plasma pushing and magnetic field penetration. The convective skin effect is induced by a density gradient normal to the direction of shock propagation. We identify a characteristic parameter, the ratio of the ion skin depth to the spatial scale of the density gradient. When this parameter is small we recover the snowplow motion. This motion is characterized by large compression, equal shock and ion velocities, and equal partition of energy between ion-directed kinetic energy and Joule dissipated energy. At the opposite limit, when the characteristic parameter is large, we recover the convective skin penetration. The ion motion is then small relative to the shock velocity, and, as a result, the ion kinetic energy is small relative to the dissipative energy. The plasma compression is small as well.

In Sec. II the model is presented. In Sec. III the two opposite limits, the snowplow motion and the convective skin penetration, are discussed.

## II. THE MODEL

The governing equations in our model are the continuity equation

$$\frac{\partial n}{\partial t} + \nabla\cdot n\mathbf{U} = 0, \tag{1}$$

the momentum balance equation

$$Mn\frac{d\mathbf{U}}{dt} = \frac{\mathbf{J}\times\mathbf{B}}{c}, \tag{2}$$

Ampère's law

$$(4\pi/c)\mathbf{J} = \nabla\times\mathbf{B}, \tag{3}$$

Faraday's law

$$-\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \nabla\times\mathbf{E}, \tag{4}$$

and Ohm's law

$$\mathbf{E} + [(\mathbf{U}\times\mathbf{B})/c] = \eta\mathbf{J} + [(\mathbf{J}\times\mathbf{B})/enc]. \tag{5}$$

Here $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, $\mathbf{J}$ is the current, $\mathbf{U}$ is the mass velocity, $n$ is the plasma density, $e$ and $M$ are the ion charge and mass, $\eta$ is the resistivity, and $c$ is the velocity of light in vacuum. Also $d/dt\equiv[(\partial/\partial t) + \mathbf{U}\cdot\nabla]$ is the convective derivative. In Ampère's law we neglected the displacement current and therefore quasineutrality is preserved. Equations (1)–(5) are the resistive magnetohydrodynamics (MHD) equations with the inclusion of the Hall field and with the neglect of the plasma pressure. The simultaneous fast magnetic field penetration and large electron heating is studied elsewhere.[11] Here, for simplicity, the plasma is assumed cold and the magnetic field energy, which is dissipated as a result of the resistivity, is assumed not to heat the plasma.

We examine the penetration of a magnetic field into the plasma. The configuration is shown in Fig. 1. The magnetic field has a $z$ component and the penetration is into a plasma of a nonuniform density $n = n_-(y)$. The parameter $L_y$ is a characteristic value of the quantity $\{(d/dy)\ln[n_-(y)]\}^{-1}$. The characteristic scale length, on which the quantities change in the $x$ direction inside the shock front, is $L_x$. Its value will be found later. We assume that

  

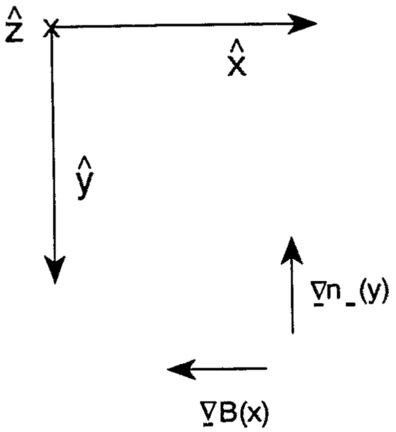


FIG. 1. The geometry. The plasma density in the shock upstream is weakly nonuniform in the $y$ direction. The shock propagates in the $x$ direction. The magnetic field has a $z$ component only.

$$\epsilon \equiv L_x/L_y \ll 1, \tag{6}$$

that $J_z = U_z = 0$, and that $J_x/J_y$ and $U_y/U_x$ are $\theta(\epsilon)$. To leading order Faraday's law becomes

$$\frac{\partial B}{\partial t} = -\frac{\partial}{\partial x}(U_x B) + \frac{c^2\eta}{4\pi}\frac{\partial^2 B}{\partial x^2} - \frac{c}{8\pi e}\left\{\frac{1}{n}, B^2\right\}. \tag{7}$$

Here

$$\{f,g\} \equiv \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

Equations (1)–(3) are approximated as

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nU_x) = 0, \tag{8}$$

$$Mn\left(\frac{\partial}{\partial t} + U_x\frac{\partial}{\partial x}\right)U_x = -\frac{\partial}{\partial x}\left(\frac{B^2}{8\pi}\right). \tag{9}$$

We look for shock solutions in which all the variables depend on $\zeta \equiv x - \lambda(y)t$ only. Equation (8) yields the relation

$$n = \lambda n_-/(\lambda - U_x). \tag{10}$$

We denote by the subscript minus (plus) the values in the shock upstream (downstream). Equations (8) and (9) yield

$$Mn_-\lambda U_x = B^2/8\pi. \tag{11}$$

Equation (7) becomes

$$\frac{c^2\eta}{4\pi}\frac{d^2B}{d\zeta^2} = -\lambda\frac{dB}{d\zeta} + \frac{d}{d\zeta}\left(\frac{B^3}{8\pi Mn_-\lambda}\right) + \frac{c}{8\pi e}\left[\frac{\partial}{\partial y}\left(\frac{1}{n_-}\right) - \frac{B^2}{8\pi}\frac{\partial}{\partial y}\left(\frac{1}{Mn_-^2\lambda^2}\right)\right]\frac{dB^2}{d\zeta}. \tag{12}$$

By integrating this equation, we obtain

$$\frac{c^2\eta}{4\pi}\frac{dB}{d\zeta} = -\lambda B + \frac{B^3}{8\pi Mn_-\lambda} + \frac{c}{8\pi e}\frac{\partial}{\partial y}\left(\frac{1}{n_-}\right)B^2 - \frac{c}{128\pi^2 e}\frac{\partial}{\partial y}\left(\frac{1}{Mn_-^2\lambda^2}\right)B^4. \tag{13}$$

At the snowplow limit we retain the first two terms on the right-hand side. When the convective skin effect is dominant we retain the first and the third term. We therefore neglect for simplicity the fourth term. The governing equation, in a nondimensional form, becomes

$$\frac{db}{d\xi} = b^3 + \frac{\lambda v_c}{v_{sp}^2}b^2 - \frac{\lambda^2}{v_{sp}^2}b. \tag{14}$$

Here $b \equiv B/B_+$, $v_{sp} \equiv B_+/(8\pi Mn_-)^{1/2}$ is the snowplow velocity, $v_c \equiv (cB_+/8\pi e)(\partial/\partial y)(1/n_-)$ is the convective skin shock velocity, and the coordinate is $\xi \equiv 4\pi v_{sp}^2\zeta/c^2\eta\lambda$. The normalization process requires that limits be placed on the allowable smallness of $\eta$. This will be addressed later in this work. It is readily seen that the shock velocity is

$$\lambda = [v_c + (v_c^2 + 4v_{sp}^2)^{1/2}]/2. \tag{15}$$

The parameter, which determines which shock is dominant, the convective skin effect, or the snowplow, is

$$R \equiv 2v_{sp}/v_c \cong 2L_y/(c/\omega_{pi}). \tag{16}$$

The convective skin effect is dominant if the density varies on a scale length smaller than the ion skin depth.

## III. THE TWO LIMITS

When $R \ll 1$ we obtain that $U_x \cong v_{sp}^2/v_c \ll \lambda \cong v_c$. The plasma motion is then much slower than the current channel motion and the plasma compression is small. If, on the other hand, $R \gg 1$, the plasma velocity downstream is $U_x \cong v_{sp}$, and the compression is very large $n_+/n_- \gg 1$. We note that the parameter $R$ can be written as

$$R = (8\pi/c^2)[v_{sp}L_y/(B_+/2n_-ec)].$$

In this form $R$ plays a role equivalent to the magnetic Reynolds number. The Hall term $B_+/2n_-ec$ is then equivalent to the resistivity and $L_y$ to the characteristic length in the usual magnetic Reynolds number.

Let us discuss the energy flow. The rate that energy goes into building the magnetic field energy in the plasma is $Q_B = (B_+^2/8\pi)\lambda$, while the rate that energy goes into ion kinetic energy is

$$Q_I = \tfrac{1}{2}n_-MU_x^2\lambda = B_+^4/2(8\pi)^2Mn_-\lambda.$$

The dissipated energy is

$$Q_H = \int_{-\infty}^{\infty} d\zeta\,\eta\frac{c^2}{(4\pi)^2}\left(\frac{dB}{d\zeta}\right)^2 = \frac{B_+^2}{4\pi}\left(\frac{\lambda}{2} - \frac{v_{sp}^2}{4\lambda} - \frac{v_c}{3}\right). \tag{17}$$

In the snowplow limit $R \gg 1$,

$$Q_B = (B_+^2/8\pi)v_{sp},$$
$$Q_I = (B_+^2/16\pi)v_{sp}, \quad (18)$$
$$Q_H = (B_+^2/16\pi)v_{sp}.$$

The magnetic field energy flux is equally divided between the magnetic field energy and the plasma energy. The part that goes to the plasma is equally divided between ion-directed kinetic energy and heat.

In the convective skin effect limit $R \ll 1$ we obtain

$$Q_B = (B_+^2/8\pi)v_c,$$
$$Q_I = (B_+^2/16\pi)(v_{sp}^2/v_c) \cong 0, \quad (19)$$
$$Q_H \cong (B_+^2/24\pi)v_c.$$

The magnetic field energy that goes to the ions is negligible. A quarter of the magnetic field energy flux goes to heating. As we have shown elsewhere,[11] the gradient in the flux of electron thermal energy further increases the electron thermal energy so that the magnetic field energy density and the electron thermal energy density in the shock downstream become equal.

Finally, we integrate Eq. (14), and obtain

$$e^{D(1+D)\xi} = [(1/b) - 1]^D[1 + (D/b)], \quad (20)$$

where $D \equiv (1/R^2)[1 + (R^2+1)^{1/2}]^2$. In the snowplow limit, $R \gg 1$ and $D = 1$. The shock structure is then

$$B^2 = B_+^2/[1 + \exp(8\pi v_{sp}\xi/c^2\eta)], \quad (21)$$

and the shock thickness is $L_x = c^2\eta/8\pi v_{sp}$. In the convective skin effect limit $R \ll 1$ and $D \cong 4/R^2$. Equation (20) is then approximated as $\exp(D\xi) = (1-b)/b$, and the shock structure becomes

$$B = B_+/[1 + \exp(4\pi v_c \xi/c^2\eta)]. \quad (22)$$

The shock thickness is $L_x = c^2\eta/4\pi v_c$.

For consistency we require that $L_x \ll L_y$. This last inequality results in the condition

$$B_+/2n_-ec \gg \eta. \quad (23)$$

The Hall term $B_+/2n_-ec$ has to be larger than the collisional resistivity. We note that the Hall term, which has the units of resistivity, is nevertheless nondissipative, and that only the collisional resistivity causes the dissipation.[9,11]

Another limitation placed on the solution by the 1-D approximation is that the solution is correct only for times $t$ that obey $t\, d\lambda/dy \ll 1$. Our convective skin effect solution therefore holds only for times shorter than the ion cyclotron period, and the 1-D shock propagates along a distance shorter than $L_y$.

In our model we have neglected the electron inertia. This is justified as long as the thickness of the shock layer is larger than the electron skin depth. In the snowplow limit, the condition is

$$\eta/(B_+/2n_-ec) > [8(m/M)]^{1/2}, \quad (24)$$

where $m$ is the electron mass.

In the convective skin effect limit the condition is

$$\eta/(B_+/2n_-ec) > (c/\omega_{pe})/L_y. \quad (25)$$

When the collision frequency $\nu(\equiv n_-e^2\eta/m)$ equals the critical collision frequency $\nu_{cr}$,

$$\nu_{cr} \equiv v_c\omega_{pe}/c, \quad (26)$$

inequality (25) ceases to be valid, and the time between collisions $\nu^{-1}$ equals the time that an electron spends in the shock front. Kalda and Kingsep have shown that when the collision frequency is smaller, the electron inertia introduces oscillatory structure into the shock front with a period on the order of collisionless skin depth.[7] Although we do not study the case of low collision rate here, a few comments are in order. The thermalization of the electron kinetic energy in the low collision rate case is achieved through the existence of the oscillatory front, which is of a thickness on the order of $v_c/\nu$. Therefore, the electron spends one collision time $\nu^{-1}$ in the shock front. The thickness of the current layer decreases monotonically with decreasing $\nu$ only for $\nu > \nu_{cr}$. When $\nu$ is smaller than $\nu_{cr}$, the thickness of the current layer increases with decreasing $\nu$. The lower limit on the collision rate is found by requiring that for a shock to exist the shock layer should be smaller than the length that the shock propagates, a length that is smaller than $L_y$. The collision frequency should thus satisfy

$$\nu > v_c/L_y. \quad (27)$$

Therefore, shock solutions due to the convective skin effect with an oscillatory front exist for resistivities that satisfy

$$\frac{c/\omega_{pe}}{L_y} > \frac{\eta}{B_+/2n_-ec} > \left(\frac{c/\omega_{pe}}{L_y}\right)^2. \quad (28)$$

When the resistivity is larger,

$$1 > \eta/(B_+/2n_-ec) > (c/\omega_{pe})/L_y, \quad (29)$$

the shock is not oscillatory and is described by Eq. (22).

Figure 2 summarizes the domains of validity of our model in the plane of the normalized scale length $L_y/(c/\omega_{pi})$ and the normalized resistivity $\eta/(B_+/2n_-ec)$. Curve 1 is $(c/\omega_{pe})/L_y$ and curve 2 is $[(c/\omega_{pe})/L_y]^2$. The convective skin effect is valid in domains $A$ [Eq. (29)] and $B$ [Eq. (28)], and the snowplow in domain $C$. In domain $A$ the shock is monotonic as described in this paper and in domain $B$ it is oscillatory. The magnetic field evolution in domain $D$ (the collisionless case) requires additional study.

In the convective skin penetration the energy flows in from the $y$ direction. In experiments such as the plasma opening switch, conductors are present at the boundaries at the $y$ direction. A recent two-dimensional analysis,[9] which took into account the conductors at the $y$ direction, showed that the magnetic field evolution in the plasma far from the conductors is described well by the 1-D approximation.

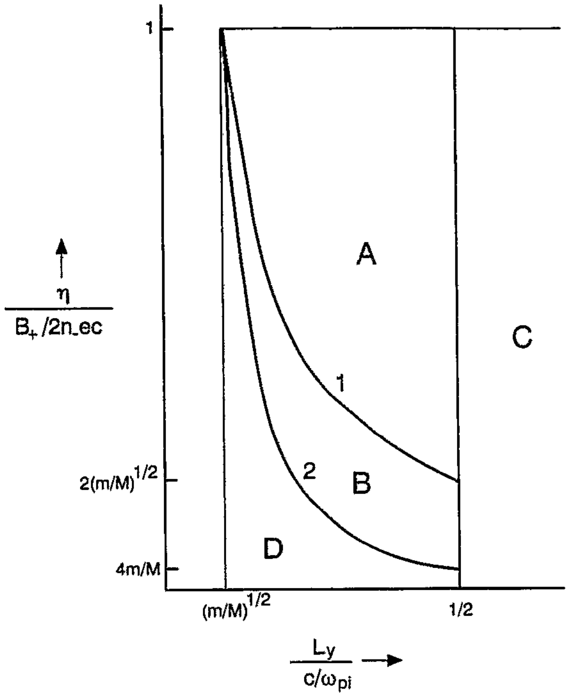


FIG. 2. The domains of validity of our model. The convective skin effect occurs in domain $A$ (monotonic shock) and in domain $B$ (oscillatory shock). The snowplow occurs in domain $C$. Here $m/M = 0.01$.

In summary, the classical snowplow motion and the convective skin effect were derived as two opposite limits of a unified model. If the dissipated energy were to increase the internal energy of the plasma, and an appropriate energy equation were used, we would find shocks with finite compression instead of the infinite compression found in the snowplow. This case will be studied in the future.

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