Evolution of a magnetic field and plasma pushing in the presence of a parallel magnetic field

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The evolution in the plasma of a magnetic field that is fast-rising at the plasma boundary, and the simultaneous pushing of the plasma by that magnetic field, are studied for the case that a parallel magnetic field is present in the plasma. It is shown that initially the magnetic field propagates into the plasma in the form of a whistler wave. The magnetic field evolution is then governed by the Schrödinger equation for a free particle, as described in the previous simplified analysis [Phys. Fluids B **3**, 1546 (1991)]. Later, the gas-dynamics shock propagation exceeds the magnetic field propagation. If the magnetic field in the plasma is initially oblique and not parallel, a quasiperpendicular fast (super-Alfvénic) shock propagates in the plasma, following a whistler precursor. The width of the current channel is on the scale of the ion skin depth.

I. INTRODUCTION

Plasma pushing by a magnetic pressure is a dominant process in magnetically driven shock waves,[1] Z pinches,[2] theta pinches,[3] plasma opening switches,[4] magnetically insulated ion diodes,[5] and other plasma devices. In a common configuration, the magnetic field is parallel to the plasma–vacuum boundary outside the plasma, while inside the plasma its magnitude drops to zero across a narrow current layer. The width of the current layer could have an important effect on the plasma dynamics. In a collisional plasma this width is usually determined by the plasma resistivity. The presence of a uniform magnetic field normal to the plasma boundary and parallel to the direction of magnetic field propagation significantly increases the width of the current channel. In a collisionless plasma the characteristic width of the current channel becomes the ion skin depth rather than the electron skin depth.

At the early stage, the magnetic field propagation into the plasma, in the course of formation of the current channel and the pushing of the plasma by the magnetic pressure, occur simultaneously. We have recently studied such a current channel formation along a parallel magnetic field, when a fast-rising magnetic field is applied at the plasma boundary.[6] We were interested in the time before the plasma is compressed and a shock is formed. By assuming that the process was so fast that the plasma ions were immobile we have shown that the magnetic field propagates into the plasma in the form of a dispersive whistler wave. In the present paper we study the current channel formation along a parallel magnetic field without neglecting the plasma motion. For a fast-rising magnetic field we show that initially the magnetic field does indeed propagate as a whistler wave and its evolution is well described by the Schrödinger equation for a free particle, in agreement with our earlier simplified analysis.[6] The plasma pushing is shown to become dominant only later. The purpose of this paper, therefore, is to describe the transition from a dominant magnetic field propagation as a whistler wave, which we have previously studied,[6] to a dominant plasma pushing in the form of a shock, as is known in shock physics. In short-duration plasmas, when the characteristic time is shorter than the ion-cyclotron period, this transition time could be comparable to the duration time of the whole process.

We employ the one-dimensional (1-D) ideal magnetohydrodynamics (MHD) model, modified by the inclusion of the Hall term in Ohm's law and artificial viscosity at the shock front. The inclusion of the Hall term extends the validity of the MHD model to times shorter than the ion-cyclotron period. Because of its dispersive nature, the Hall term is dominant when the fast-rising magnetic field is applied. It is the Hall field that enables the magnetic field to propagate in the form of a whistler wave. The magnetic field is frozen into the electron fluid in our model of zero resistivity.

As the current channel broadens, the velocity of this broadening decreases. The magnetic pressure acts as a piston and generates a gas-dynamic shock in the plasma. The propagation of the shock and the accompanying plasma motion gradually become the dominant processes.

The present analysis describes the transient process of magnetic field propagation and plasma pushing along a background parallel magnetic field in an idealized one-dimensional semi-infinite plasma. The time scale, the ion-cyclotron period, is characteristic of several pulsed-power plasma devices. The understanding of the mechanism of current channel broadening could be of practical importance. However, the effect of the inclusion of a parallel magnetic field in any such device could only be appreciated through a more-detailed analysis that includes two-dimensional (2-D) effects as well as finite dimensions of the particular system, sheaths at boundaries, etc.

In Sec. II we write the governing equations. In Sec. III we present numerical examples that demonstrate the early field propagation and the developing gas-dynamic shock. In Sec. IV we show the fast shock propagating in an initially magnetized plasma and the whistler precursor in its front.

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II. THE GOVERNING EQUATIONS

We consider a plasma that is initially of a uniform density ρ_0 and that is propagated by a magnetic field of a characteristic magnitude B_0. The 1-D MHD equations, with the addition of the Hall term, are the continuity equation

$$\frac{d\rho}{dt}+\rho\frac{\partial u_x}{\partial x}=0, \tag{1a}$$

the momentum equation

$$\frac{du_x}{dt}+\frac{1}{\rho}\frac{\partial}{\partial x}\left(p+q+\frac{|b_\perp|^2}{2}\right)=0, \tag{1b}$$

$$\frac{du_\perp}{dt}=\frac{b_x}{\rho}\frac{\partial b_\perp}{\partial x}, \tag{1c}$$

the energy equation

$$\frac{d\epsilon}{dt}=\frac{(p+q)}{\rho^2}\frac{d\rho}{dt}, \tag{1d}$$

and Faraday's law

$$\frac{db_\perp}{dt}=b_x\frac{\partial u_\perp}{\partial x}-b_\perp\frac{\partial u_x}{\partial x}-ib_x\frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial b_\perp}{\partial x}\right). \tag{1e}$$

The Hall term is the last term on the right-hand side of Eq. (1e). We have used normalized quantities, where ρ is the density divided by ρ_0, **b** is the magnetic field divided by B_0, x is the coordinate divided by $x_0\equiv(c^2M^2/4\pi\rho_0e^2)^{1/2}$ (e and M are the ion charge and mass and c is the velocity of light in vacuum), and t is the time divided by $t_0\equiv(cM/eB_0)$. The normalized velocity **u** is the velocity divided by $u_A\equiv x_0/t_0$. Thus u_A is the Alfvén velocity calculated with the intensity of the magnetic piston and the mass density of the unperturbed plasma, while x_0, the ion skin depth, is also the ion Larmor radius calculated with the intensity of the magnetic piston and the Alfvén velocity u_A. Also, p and q are the pressure and viscosity divided by $B_0^2/4\pi$, and ϵ is the internal energy divided by u_A^2. We write the transverse components of the magnetic field and of the velocity in a complex form $u_\perp\equiv u_z+iu_y$, $b_\perp\equiv b_z+ib_y$, and define $d/dt\equiv\partial/\partial t+u_x(\partial/\partial x)$. This fairly standard form of the equations is similar to the form used, for example, by Kennel *et al.*[7] However, since we allow shock formation, the viscosity in our model is not zero and the entropy is not constant. For simplicity we choose a polytropic equation of state $p=A\rho^\gamma$, and assume the internal energy to be of the form $\epsilon=p/\rho(\gamma-1)$. We can then replace the energy equation by the equation

$$\frac{dA}{dt}-(\gamma-1)\rho^{-\gamma-1}q\frac{d\rho}{dt}=0. \tag{2}$$

For the numerical calculation it is convenient to employ Lagrangian space, $\xi\equiv\int_0^x dx'\,\rho(x')$, and time, $\tau\equiv t$, coordinates. Equations (1a)–(1c), (1e), and (2) become

$$\frac{\partial V}{\partial\tau}=\frac{\partial u_x}{\partial\xi}, \tag{3a}$$

$$\frac{\partial u_x}{\partial\tau}+\frac{\partial}{\partial\xi}\left(p+q+\frac{|b_\perp|^2}{2}\right)=0, \tag{3b}$$

$$\frac{\partial u_\perp}{\partial\tau}=b_x\frac{\partial b_\perp}{\partial\xi}, \tag{3c}$$

$$\frac{\partial A}{\partial\tau}+(\gamma-1)V^{\gamma-1}q\frac{\partial V}{\partial\tau}=0, \tag{3d}$$

and

$$\frac{\partial(b_\perp V)}{\partial\tau}=b_x\frac{\partial u_\perp}{\partial\xi}+ib_x\frac{\partial^2 b_\perp}{\partial\xi^2}. \tag{3e}$$

Here $V\equiv1/\rho$ is the specific volume. In the next section we solve these equations for the particular problem that a magnetic field is switched on at the plasma boundary.

III. NUMERICAL EXAMPLES

We study the idealized problem of a semi-infinite uniform plasma of initial pressure $p_0=0$ that is located at $x\geqslant0$. A uniform constant-in-time magnetic field b_x is present in the plasma. At $t=0$ a magnetic field $b=1$ is switched on at $x\leqslant0$. We solve Eqs. (3) for the evolution of the magnetic field and of the plasma. We assume that the motion of the plasma and the propagation of the magnetic field do not change the intensity of the magnetic field at the plasma–vacuum boundary at $x\leqslant0$ and therefore $b(\xi=0,\tau>0)=1$. We also assume that $\gamma=2$ and that the (small) artificial viscosity is of the form[8]

$$q=\begin{cases}\dfrac{a^2h^2}{V}\left(\dfrac{\partial u_x}{\partial\xi}\right)^2, & \text{if } \dfrac{\partial u_x}{\partial\xi}<0,\\[2ex] 0, & \text{if } \dfrac{\partial u_x}{\partial\xi}\geqslant0.\end{cases} \tag{4}$$

This Lagrangian form of the equations, including the electron inertia that we omit, was used by Morton[9] to study the evolution of perpendicular shocks. Figures 1(a)–1(d) show the amplitude of the magnetic field $|b|$, the plasma velocity u_x, and the specific volume V vs ξ for times $t=0.1, 1, 10, 50$, respectively. For comparison, Fig. 2 shows u_x and V for the time $\tau=1$ when the parallel magnetic field is absent, $b_x=0$. The propagation of the magnetic field is dominant at first and is not influenced by the plasma motion. At later times the propagation slows down because of the dispersive nature of the whistler wave, and the shock front catches up with the magnetic field [Fig. 1(c)]. At even later times [Fig. 1(d)], the shock velocity exceeds the velocity of the magnetic field propagation. The magnetic field is then confined to the volume of the compressed plasma in the shock downstream. The plasma at the shock downstream is not uniform when the magnetic field propagates and at the plasma–vacuum boundary there is a regime of rarefaction.[1] As expected, at later times, when the shock front by far exceeds the magnetic field propagation [Fig. 1(d)], the jump relations satisfy the usual gas-dynamics Hugoniot relations.[10] The compression ratio, for example, is then $\frac{1}{3}[=(\gamma-1)/(\gamma+1)]$.

In our previous paper[6] we have studied the magnetic field propagation at such early times that the ions do not move. At these early times $V=1$ and $\mathbf{u}=0$. Equation (3e) for the magnetic field then becomes the Schrödinger equation for a free particle[6]

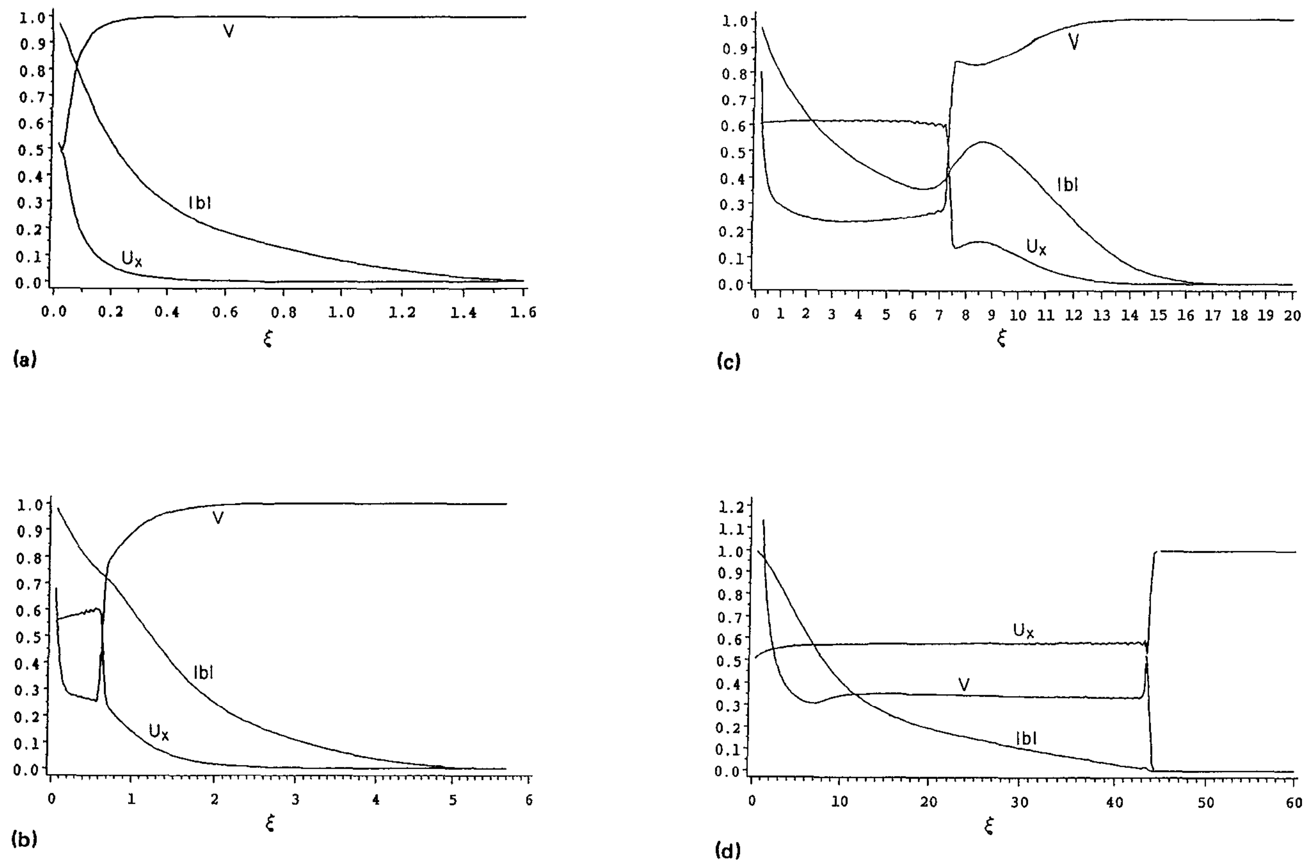

FIG. 1. The intensity of the magnetic field $|b|$, the longitudinal velocity u_x, and the specific volume V versus the Lagrangian coordinate ξ, for various times: (a) $\tau = 0.1$; (b) $\tau = 1$; (c) $\tau = 10$; (d) $\tau = 50$. The parameters are $b_x = 0.1$, $\gamma = 2$, and the pressure and perpendicular magnetic field upstream zero.

$$\frac{\partial b_\perp}{\partial \tau} = ib_x \frac{\partial^2 b_\perp}{\partial \xi^2}. \tag{5}$$

We call $MB_0 b_x/(\rho ec)$ the Hall resistivity, because the magnetic field evolution-in-time scales with this nondissipative "resistivity" as it scales with the collisional resistivity in the usual diffusion with a real diffusion coefficient. The solution of this equation with the initial-boundary conditions of this section is

$$b_\perp = 1 - (1 - i)\{C[\xi/(2\pi b_x \tau)^{1/2}] - iS[\xi/(2\pi b_x \tau)^{1/2}]\}, \tag{6}$$

where C and S are Fresnel integrals.[11] Figures 3(a)–3(b) compare the magnetic field components $b_z(\mathrm{Re}\, b)$ and $b_y(\mathrm{Im}\, b)$ as a function of ξ for the times $\tau = 0.1$, 1, respectively, as found from the numerical solutions of Eqs. (3) versus their forms given by Eq. (6). At the earlier time the correspondence is very good and the predictions of our approximate model[6] are fairly accurate. As expected, at the later times the propagation of the magnetic field is modified by the plasma motion and the magnetic field profile deviates from that predicted by (6).

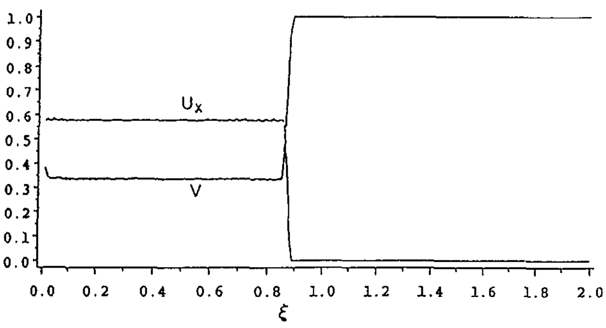

FIG. 2. The velocity u_x and the specific volume V vs ξ at $\tau = 1$. The parameters are as in Fig. 1 except that $b_x = 0$.

IV. GENERATION OF A WHISTLER PRECURSOR

In the previous sections we described a gas-dynamics shock that was generated by switching a magnetic field at the boundary of the plasma. In this section we show a numerical

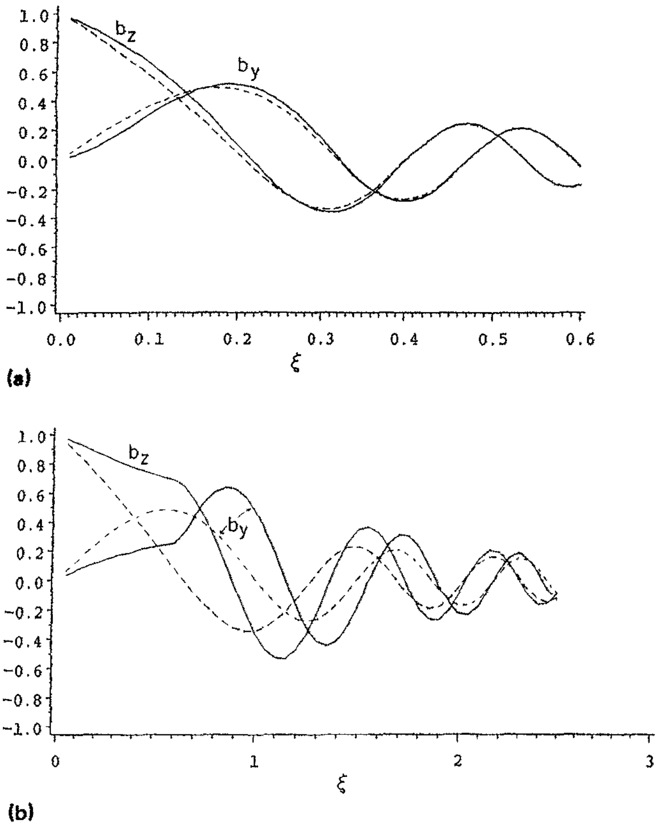

FIG. 3. The magnetic field components b_z and b_y vs ξ at (a) $\tau = 0.1$ and (b) $\tau = 1$. The parameters are as in Fig. 1. The solid lines show the solutions of the full equations and the dashed lines show the approximate form (6).

solution for the familiar case that the magnetic field in the plasma is oblique, rather than parallel, and the shock transition includes compression of the magnetic field within the plasma. The presence of a finite parallel magnetic field b_x results in oscillations in the shock front. This oscillating shock precursor is generated by the dispersive Hall term (solving the MHD equations without the Hall term results in nonoscillatory sharp transition).

Figure 4 shows such generated quasiperpendicular shock, where the magnetic field is $\mathbf{b} = 0.7\hat{e}_z + 0.1\hat{e}_x$, and the pressure upstream is zero. The pressure balance at the vacuum–plasma downstream boundary is

$$b_{\text{ext}}^2/2 = p_1 + b_1^2/2, \tag{7}$$

where subscript 1 denotes downstream values. Inclusion of the Hall term in the MHD equations does not change the Hugoniot jump relations. The effect of the dispersive Hall term is to generate the whistler precursor at the front of this quasiperpendicular fast (super-Alfvénic) shock.[12] In the numerical example: $b_x = 0.1$, $V_0 = 1$, $p_0 = 0$, $\mathbf{u}_0 = 0$, $b_{z0} = 0.7$. The values at the shock downstream are $b_1 \cong 1$, $v_1 = 0.73$, $p_1 \cong 0$, $u_{x1} = 0.24$, and the shock velocity is 0.9, in agreement with the Hugoniot relations[12] and (7).

We look for stationary wave solutions for Eqs. (3) that depend on the variable $\eta \equiv \xi - \lambda\tau$, where λ is a constant.

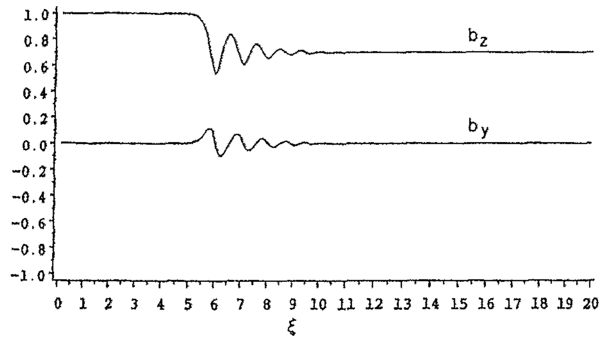

FIG. 4. The magnetic field components b_z and b_y vs ξ at $\tau = 1$, when b_z upstream is 0.7. The rest of the parameters are as in Fig. 1.

The stationary equations in the case of zero viscosity and pressure combine to give

$$ib_x \frac{db_\perp}{d\eta} = \frac{(|b_\perp|^2 - b_0^2)}{2\lambda} b_\perp - \left(\lambda - \frac{b_x^2}{\lambda}\right)(b_\perp - b_0). \tag{8}$$

A similar equation has been extensively studied, for example, in relation to intermediate shock waves.[13] We can view the evolution of the magnetic field governed by the above equation as a motion of a massless particle in the (b_z, b_y) plane (η is the time) under the influence of a uniform magnetic field b_x and a potential[13]

$$\psi = \frac{(|b_\perp|^2 - b_0^2)^2}{8\lambda} - \left(\lambda - \frac{b_x^2}{\lambda}\right)\frac{|b_\perp - b_0|^2}{2}, \tag{9}$$

where

$$b_x \frac{db_z}{d\eta} = \frac{\partial\psi}{\partial b_y}, \tag{10}$$

$$-b_x \frac{db_y}{d\eta} = \frac{\partial\psi}{\partial b_z}. \tag{11}$$

The motion is periodic and results in the whistler oscillations.

The magnetic field evolves along equipotential curves and the Hall term is the source of the oscillations around the fixed point. The dissipation damps the oscillations and results in a shock transition.

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