The lower-hybrid drift instability in a slab geometry

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The lower-hybrid drift instability is studied within a model problem with some of the features that characterize the anode plasma in the magnetic insulated diode experiment at the Weizmann Institute (Phys. Rev. A, in press). The spatial dependence of the amplitude of the linear electrostatic perturbations is calculated. First the equilibrium is determined by selecting distribution functions that depend on the constants of motion in a collisionless plasma and by imposing quasineutrality. In the particular equilibrium studied, the electric field, temperature, and drift velocities are uniform and the density decreases exponentially. The Vlasov–Poisson equations are then linearized assuming magnetized electrons and unmagnetized ions. The governing equation is a second-order ordinary differential equation for the perturbed electrostatic potential. A dispersion relation is derived and analytical solutions are written for the eigenfunctions. The growth rate of each mode is determined by the plasma parameters at its respective turning point. Since the only plasma parameter that is not uniform is the density, the growth rates of the different eigenmodes are similar when the roots of the local dispersion relation depend only weakly on the density. Since the distance between successive turning points is usually very small, it is concluded that for the chosen equilibrium the perturbations will grow uniformly across the plasma.

I. INTRODUCTION

It has recently been suggested that the lower-hybrid drift instability causes the anomalous diffusion and consequently the fast expansion of a plasma into the gap in the magnetic insulated diode experiment at the Weizmann Institute.[1] The diffusion coefficients depend on the wave amplitude, and the spatial dependence of those coefficients is related to the spatial dependence of the wave amplitude. The lower-hybrid drift instability has been studied extensively in the past.[2–19] Our purpose is to study the spatial dependence of the wave amplitude in a model problem with some of the features that characterize the Weizmann experiment. To that end we address the nonlocal problem, taking into account the finite dimensions of the plasma. Figure 1 is a schematic of the magnetic insulated diode experiment at the Weizmann Institute. The distance between the anode and the cathode is much smaller than the dimensions of the electrodes. Therefore in our model problem all time-independent quantities are approximated to depend on x only, the direction perpendicular to the anode and cathode planes. The plasma is confined to the anode by an external uniform magnetic field, which lies on the anode plane, parallel to the z axis. The nonuniformity of the plasma is accompanied by a time-independent electric field in the x direction and drift velocities in the y direction.

The equilibrium distribution functions of the ions and of the electrons are solutions of the Vlasov equation in the presence of the equilibrium electric and magnetic fields. These solutions are functions of the three constants of motion: the total perpendicular energy, the y canonical momentum, and the velocity component parallel to the magnetic field. Following Davidson[8] we assume that the functional forms of the distribution functions of the ions and of the electrons are the same, and then, by imposing quasineutrality, we derive the form of the equilibrium electric field. For the particular chosen form of the distribution functions, the density is exponentially decreasing, and the equilibrium electric field, drift velocities, and temperatures of the ions and the electrons are all x independent. Having specified the equilibrium we then perform a linear stability analysis. The linearized Vlasov–Poisson equations are reduced to a second-order differential equation for the perturbed electrostatic potential. We have assumed that the electrons are strongly magnetized and that the ions are unmagnetized. An analytical solution is given to the governing equation and the boundary conditions yield a dispersion relation. The eigenfunctions of the equation describe the x profile of the perturbation and the imaginary part of the eigenvalue determines the growth rate of the instability of the mode. The forms of the eigenfunctions are determined by kd, k being the wavenumber of the perturbation in the y direction, and d the plasma thickness. Solving the governing equation yields a set of eigenfunctions and a corresponding set of values of some parameter g, for each value of kd. The parameters g and kd determine the location of the turning point of the eigenfunction, and the growth rate of the instability is found by solving the usual local dispersion relation at the turning point. This turning point is

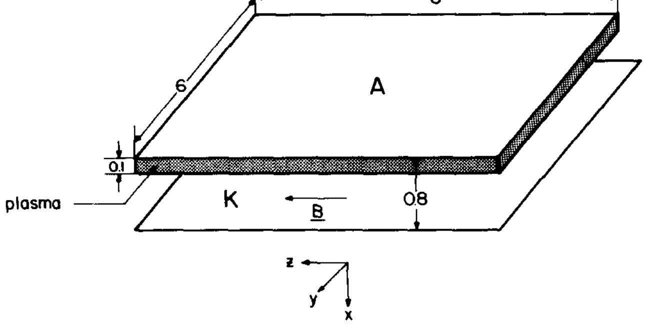

FIG. 1. Schematic of the magnetic insulated diode experiment at the Weizmann Institute. Dimensions are in centimeters.

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shown to be close to the first maximum and first zero of the eigenfunction. Thus the growth rates of the instability of the various modes differ by the values of the plasma parameters at the various turning points. Since the only parameter whose equilibrium value varies across the plasma is the density, the stability of the various modes differs by the plasma density at the turning point.

For the solution of the nonlocal problem we may now use the extensive parameter studies that already exist for the local dispersion relation. For the parameters of the experiment at the Weizmann Institute[1] we use the approximate results of Davidson and Gladd.[5] The growth rate of the instability is very weakly dependent on the plasma density and therefore many modes will grow with a similar rate. Since the distance between successive turning points is very small, and since the amplitude of each mode is maximal near its turning point, the level of fluctuations will be uniform across the plasma. The diffusion coefficients of the anomalous diffusion are also expected to be uniform.

In Sec. II we derive the governing equation. In Sec. III we derive analytical solutions to the nonlocal problem and relate them to the solutions of the local dispersion relation. In light of the solutions of the nonlocal problem, the local solutions can be clearly interpreted. Numerical examples of the Weizmann experiment are presented and discussed in Sec. IV.

II. DERIVATION OF THE GOVERNING EQUATION

Let us consider a plasma with an equilibrium that is x dependent only, and that is immersed in a uniform magnetic field of the form

$$\mathbf{B}_0 = B_0 \hat{e}_z . \tag{1}$$

In choosing the equilibrium, our analysis follows Davidson's analysis for systems of cylindrical symmetry.[8] We also use some of his notation. We assume that the equilibrium electric field has an x component only. Three single-particle constants of motion are the axial velocity v_z, the perpendicular Hamiltonian $H_\perp$, and the canonical momentum p_y, where

$$H_\perp = (m_j/2)(v_x^2 + v_y^2) + e_j \Phi_0(x) \tag{2}$$

and

$$p_y = m_j (v_y + \epsilon_j \Omega_j x) . \tag{3}$$

Here e_j and m_j are the charge and mass of a particle of species j, $\Phi_0(x)$ is the electrostatic potential, $\epsilon_j = \text{sgn}\, e_j$, and $\Omega_j = |e_j| B_0/(m_j c)$ is the cyclotron frequency. From here on we will study the stability of equilibria that are functions of these three constants of motion,

$$f_j^{(0)}(\mathbf{x},\mathbf{v}) = f_j(H_\perp, p_y) G_j(v_z) , \tag{4}$$

where the parallel velocity distribution satisfies the normalization condition $\int_{-\infty}^{\infty} dv_z\, G_j(v_z) = 1$. One may examine various equilibrium distribution functions of the form (4) that model different types of equilibria. Meanwhile we do not specify the form of the equilibrium.

The linearized Vlasov equation is

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{e_j}{m_j}\left(\mathbf{E}_0 + \frac{\mathbf{v}\times B_0\hat{e}_z}{c}\right)\cdot\frac{\partial}{\partial \mathbf{v}}\right]\delta\hat{f}_j(\mathbf{x},\mathbf{v},t) = \frac{e_j}{m_j}\nabla\delta\hat{\Phi}\cdot\frac{\partial}{\partial \mathbf{v}} f_j^{(0)}(\mathbf{x},\mathbf{v}) , \tag{5}$$

where $\delta\hat{f}_j$ and $\delta\hat{\Phi}$ are the perturbed distribution function and the perturbed electrostatic potential, and $\mathbf{E}_0$ is the equilibrium electric field,

$$\mathbf{E}_0 = -\frac{\partial \Phi_0}{\partial x}\hat{e}_x . \tag{6}$$

With Eq. (4) we write Eq. (5) as

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \frac{e_j}{m_j}\left(\mathbf{E}_0 + \frac{\mathbf{v}\times B_0\hat{e}_z}{c}\right)\cdot\frac{\partial}{\partial \mathbf{v}}\right]\delta\hat{f}_j(\mathbf{x},\mathbf{v},t) = \frac{e_j}{m_j}\nabla\delta\hat{\Phi}\left(m_j(\mathbf{v} - \hat{e}_z v_z)\frac{\partial f_j^{(0)}}{\partial H_\perp} + m_j \hat{e}_y \frac{\partial f_j^{(0)}}{\partial p_y} + \hat{e}_z \frac{\partial f_j^{(0)}}{\partial v_z}\right). \tag{7}$$

We substitute

$$\delta\hat{\Phi}(\mathbf{x},t) = \delta\overline{\Phi}(\mathbf{x})\exp(-i\omega t), \quad \text{Im}\,\omega > 0 , \tag{8}$$

in the right-hand side of Eq. (7) and integrate from $t' = -\infty$ to $t' = t$. Assuming the initial perturbations to be zero and noting that v_z, $\partial f_j^{(0)}/\partial H_\perp$, and $\partial f_j^{(0)}/\partial v_z$ are constant along an unperturbed particle trajectory, we obtain

$$\begin{aligned}\delta\hat{f}_j(\mathbf{x},\mathbf{v},t) = {} & \frac{e_j}{m_j}\left(\frac{\partial f_j^{(0)}}{\partial v_z} - m_j v_z \frac{\partial f_j^{(0)}}{\partial H_\perp}\right) \\ & \times \int_{-\infty}^{t} dt' \exp(-i\omega t')\frac{\partial}{\partial z'}\delta\Phi[\mathbf{x}'(t')] \\ & + e_j \frac{\partial f_j^{(0)}}{\partial H_\perp}\int_{-\infty}^{t} dt' \\ & \times \exp(-i\omega t')\mathbf{v}'(t')\cdot\nabla'\delta\overline{\Phi}[\mathbf{x}'(t')] \\ & + e_j \frac{\partial f_j^{(0)}}{\partial p_y}\int_{-\infty}^{t} dt' \\ & \times \exp(-i\omega t')\frac{\partial}{\partial y'}\delta\overline{\Phi}[\mathbf{x}'(t')] .\end{aligned} \tag{9}$$

The particle trajectories $\mathbf{x}'(t')$ and $\mathbf{v}'(t')$ satisfy

$$\frac{d}{dt'}\mathbf{x}' = \mathbf{v}' \tag{10}$$

and

$$\frac{d}{dt'}\mathbf{v}' = \frac{e_j}{m_j}\left(\mathbf{E}_0(\mathbf{x}') + \frac{\mathbf{v}'\times B_0\hat{e}_z}{c}\right). \tag{11}$$

Making use of

$$\mathbf{v}'\cdot\nabla'\delta\overline{\Phi}(\mathbf{x}') = \frac{d}{dt'}\delta\overline{\Phi}(\mathbf{x}') , \tag{12}$$

and integrating by parts, we find

$$\delta\hat{f}_j(\mathbf{x},\mathbf{v},t) = \frac{e_j}{m_j}\left(\frac{\partial f_j^{(0)}}{\partial v_z} - m_j v_z \frac{\partial f_j^{(0)}}{\partial H_\perp}\right)\int_{-\infty}^{t} dt' \times \exp(-i\omega t')\frac{\partial}{\partial z'}\delta\Phi[\mathbf{x}'(t')] + e_j\frac{\partial f_j^{(0)}}{\partial H_\perp} e^{-i\omega t}\delta\bar{\Phi}(\mathbf{x}) + i\omega e_j \frac{\partial f_j^{(0)}}{\partial H_\perp}\int_{-\infty}^{t} dt' \times \exp(-i\omega t')\delta\bar{\Phi}[\mathbf{x}'(t')] + e_j\frac{\partial f_j^{(0)}}{\partial p_y}\int_{-\infty}^{t} dt' \times \exp(-i\omega t')\frac{\partial}{\partial y'}\delta\bar{\Phi}[\mathbf{x}'(t')]\,. \quad (13)$$

We now write

$$\delta\bar{\Phi}(\mathbf{x}) = \delta\Phi(x)\exp[i(k_y y + k_z z)] \quad (14)$$

and

$$\delta\hat{f}(\mathbf{x},\mathbf{v},t) = \delta f(x,\mathbf{v})\exp[i(k_y y + k_z z - \omega t)]\,. \quad (15)$$

Equation (13) now becomes

$$\delta f(x,\mathbf{v}) = \frac{e_j}{m_j}\left(\frac{\partial f_j^{(0)}}{\partial v_z} - m_j v_z\frac{\partial f_j^{(0)}}{\partial H_\perp}\right) ik_z I + e_j\frac{\partial f_j^{(0)}}{\partial H_\perp}\delta\Phi + i\omega e_j\frac{\partial f_j^{(0)}}{\partial H_\perp} I + ik_y e_j\frac{\partial f_j^{(0)}}{\partial p_y} I\,, \quad (16)$$

where

$$I \equiv \int_{-\infty}^{0} d\tau \exp(-i\omega\tau)\delta\Phi[x'(\tau)] \times \exp\{ik_y[y(\tau)-y(0)] + ik_z[z(\tau)-z(0)]\} \quad (17)$$

and

$$\tau \equiv t' - t\,, \quad (18a)$$

$$x'(\tau) \equiv x'(t')\,. \quad (18b)$$

The particle trajectories are

$$\frac{dv_x}{dt'} = \frac{e_j}{m_j}E_0 + \epsilon_j\Omega_j v_y\,, \quad (19a)$$

$$\frac{dv_y}{dt'} = -\epsilon_j\Omega_j v_x\,, \quad (19b)$$

$$\frac{dv_z}{dt'} = 0\,. \quad (19c)$$

At this stage we specify the equilibrium distribution function that will be studied in the rest of this paper. The chosen equilibrium distribution function is

$$f_j^{(0)}(H_\perp,p_y) = \left(\frac{m_j N_j}{2\pi T_{\perp j}}\right)\exp\left(-\frac{m_j v_{yj}^2}{2T_{\perp j}}\right)\exp\left(-\frac{(H_\perp - v_{yj}p_y)}{T_{\perp j}}\right), \quad (20)$$

where N_j, $T_{\perp j}$, and v_{yj} are constants. We require quasineutrality;

$$N_{i0}(x) = N_{e0}(x) \quad (21)$$

assuming singly ionized ions. Quasineutrality is satisfied if the exponents in the distribution functions (20) are equal for the two species. Therefore, the electrostatic potential is

$$\Phi_0(x) = \frac{T_{\perp e}T_{\perp i}B_0}{(T_{\perp e}+T_{\perp i})c}\left(\frac{v_{ye}}{T_{\perp e}} - \frac{v_{yi}}{T_{\perp i}}\right)x\,. \quad (22)$$

The corresponding electric field is

$$E_0(x) = \frac{T_{\perp e}T_{\perp i}B_0}{(T_{\perp e}+T_{\perp i})c}\left(\frac{v_{yi}}{T_{\perp i}} - \frac{v_{ye}}{T_{\perp e}}\right), \quad (23)$$

and is constant. Imposing quasineutrality on the chosen forms of equilibrium distribution functions determines the form of the electric field. While Davidson's electric field varied across the cylindrical plasma,[8] the electric field in our slab plasma is uniform. One may easily verify that the equilibrium is isothermal and that $T_{\perp j}$ are the perpendicular temperatures. Similarly v_{yj} are found to be the uniform flow velocities of the electrons and the ions. The density profile of both species is

$$N(x) = N_A\exp(-x/d)\,, \quad (24)$$

where

$$d = c(T_{\perp e} + T_{\perp i})/eB_0(v_{yi} + v_{ye})\,. \quad (25)$$

Note that for the assumed form of the distribution functions all the parameters are determined by specifying B_0, $T_{\perp e}$, $T_{\perp i}$, d, and at least one of the flow velocities v_{yi} or v_{ye}. If both flow velocities are unknown, one has to assume some value for one of them. In some cases it is reasonable to assume that v_{yi} is zero. Meanwhile, we do not specify the value of v_{yi}. Defining the ratio of the flow velocities r as

$$r = v_{yi}/v_{ye}\,, \quad (26)$$

we write equivalently to (22) and (23)

$$\Phi_0(x) = [(T_{\perp i} - T_{\perp e}r)/e(1+r)](x/d)\,, \quad (27)$$

$$E_0 = (T_{\perp e}r - T_{\perp i})/e(1+r)d\,, \quad (28)$$

and

$$v_{ye} = (c/eB_0 d)(T_{\perp e} + T_{\perp i})/(1+r)\,, \quad (29a)$$

$$v_{yi} = rv_{ye}\,. \quad (29b)$$

With the equilibrium distribution functions specified, we now turn back to the perturbed distribution functions. The ions and electrons are analyzed separately, starting with the electrons. We use cylindrical coordinates $(v_\perp, \theta, v_z)$ for the velocity

$$v_\perp\cos\theta = v_x\,, \quad (30a)$$

$$v_\perp\sin\theta = v_y - v_{yj}\,. \quad (30b)$$

The unperturbed orbits are

$$y(\tau) - y(0) = -\frac{cE_0}{B_0}\tau - \frac{1}{\Omega}\left[\cos(\Omega\tau)v_\perp\cos\theta - \sin(\Omega\tau) \times \left(c\frac{E_0}{B_0} + v_y + v_\perp\sin\theta\right) - v_\perp\cos\theta\right], \quad (31a)$$

$$z(\tau) - z(0) = v_z\tau\,. \quad (31b)$$

Here we define v_E, the single-particle drift velocity, as

$$v_E \equiv -\frac{cE_0}{B_0} = \frac{c}{eB_0 d}\frac{(T_{\perp i} - rT_{\perp e})}{(1+r)}, \tag{32}$$

and v_{Dj}, the diamagnetic drift velocity, as

$$v_{Dj} \equiv \frac{c}{e_j B_0 N}\frac{\partial}{\partial x}(NT_{\perp j}) = -\frac{cT_{\perp j}}{e_j B_0 d}. \tag{33}$$

Equations (29), (32), and (33) yield

$$v_{yj} = v_E + v_{Dj}. \tag{34}$$

We could write an expression similar to (31) for $x(\tau)$. However, a major simplification of the problem is achieved by assuming that

$$\frac{\partial \delta\Phi}{\partial x}\cdot r_L \ll \delta\Phi, \tag{35}$$

where r_L is the electron Larmor radius $[r_L \equiv (2T_{\perp e}/m_e)^{1/2}/\Omega_e]$. With this assumption, $x(\tau)$ is $x(0)$ in Eq. (17) and

$$\delta\Phi[x(\tau)] = \delta\Phi[x(0)] = \delta\Phi(x). \tag{36}$$

This approximation results from neglecting finite Larmor radius effects (FLR) in the x direction. We do not, however, neglect FLR effects in the y direction, the direction of the drift velocities. In the y direction we allow spatial field variations on a scale length comparable to the electron Larmor radius. The possibility of using the approximation (36) instead of expanding $\delta\Phi$ to second order in the electron Larmor radius was mentioned by Kent and Taylor.[20] Not only do we assume that the variations of the perturbations in the x direction are slow, we even allow them to vary on the same scale length on which the equilibrium quantities vary, and, explicitly,

$$\frac{d}{dx}(\ln \delta\Phi) \leqslant \frac{d(\ln N)}{dx}. \tag{37}$$

In doing so, we expand the validity of our formalism to a parameter regime where local theory is not valid. A necessary condition for the validity of a local theory is that the logarithmic derivative of the perturbed quantities be much smaller than the logarithmic derivative of the equilibrium quantities. We do not restrict ourselves to such a condition. Our theory holds when (37) is satisfied.

In the expression for the electron-perturbed distribution function we now perform the integration along the unperturbed orbits in the standard way. We then calculate $\delta\rho_e$, the electron perturbed density,

$$\delta\rho_e = -e\int_0^{2\pi} d\theta \int_0^\infty v_\perp\, dv_\perp \int_{-\infty}^{\infty} dv_z\, \delta f_e. \tag{38}$$

Since we are interested in the lower-hybrid drift instability we assume that

$$\omega \ll \Omega_e. \tag{39}$$

Moreover, we assume that

$$k_z = 0 \tag{40}$$

following Gladd,[7] who showed that the growth rate of the instability is largest if k_z is zero. With these two last assumptions the electron-perturbed density becomes

$$\delta\rho_e(x) = -\frac{N_A e^2}{T_e}\exp\left(-\frac{x}{d}\right)\delta\Phi(x) \times\left(1 - \exp(-\zeta)S + \frac{kv_{De}}{(\omega - kv_E)}\exp(-\zeta)S\right), \tag{41}$$

where

$$S \equiv \sum_{l=-\infty}^{\infty} J_{-l}\left(\frac{k}{\Omega_e}v_{De}\right)I_l(\zeta) \tag{42a}$$

and

$$\zeta \equiv \tfrac{1}{2}k^2 r_L^2. \tag{42b}$$

We suppressed the subscript in k_y and used the equality

$$v_{De} = v_{ye} - v_E. \tag{43}$$

We note now that

$$kV_{De}/\Omega_e = \zeta/kd. \tag{44}$$

and study the case where ζ is of order 1 and kd is large. This corresponds to the case in which the wavelength of the perturbation is comparable to the electron Larmor radius, and is much smaller than the slab thickness. With this assumption S is approximated as

$$S = I_0(\zeta). \tag{45}$$

We now turn to the ions. Since the wave frequency is assumed to be much larger than the ion cyclotron frequency, we make the usual approximation that the ions are unmagnetized. The ion perturbed density thus obtained is

$$\delta\rho_i = -\frac{N_A e^2}{T_i}\exp\left(\frac{-x}{d}\right)\delta\Phi(x) \times\left[1 + \left(\frac{\omega - kv_{yi}}{kv_{\text{th}i}}\right)Z\left(\frac{\omega - kv_{yi}}{kv_{\text{th}i}}\right)\right], \tag{46}$$

where $v_{\text{th}i}$ is the ion thermal velocity $[v_{\text{th}i} \equiv (2T_i/m_i)^{1/2}]$ and Z is the plasma dispersion function.

We make the electrostatic approximation that is valid for low beta plasmas.[8] We substitute the electron and ion charge densities into Poisson's equation, and, finally, obtain

$$\frac{d^2\delta\Phi}{dx^2} - \left(k^2 + \frac{1}{\lambda_{DA}^2}\exp\left(\frac{-x}{d}\right)\left\{1 - I_0(\zeta)\exp(-\zeta) + \frac{kv_{De}}{(\omega - kv_E)}I_0(\zeta)\exp(-\zeta) + \left(\frac{T_e}{T_i}\right) \times\left[1 + \left(\frac{\omega - kv_{yi}}{kv_{\text{th}i}}\right)Z\left(\frac{\omega - kv_{yi}}{kv_{\text{th}i}}\right)\right]\right\}\right)\delta\Phi = 0. \tag{47}$$

Here λ_{DA} is the electron Debye length at the anode $[\lambda_{DA}^2 \equiv T_{\perp e}/(4\pi N_A e^2)]$. Equating the coefficient of $\delta\Phi$ in Eq. (47) to zero gives us the local dispersion relation[5] at the point x:

$$k^2 + \frac{1}{\lambda_D^2}\left\{1 - I_0(\zeta)\exp(-\zeta) + \frac{kv_{De}}{(\omega - kv_E)}I_0(\zeta)\exp(-\zeta) + \left(\frac{T_e}{T_i}\right)\left[1 + \left(\frac{\omega - kv_{yi}}{kv_{\text{th}i}}\right)Z\left(\frac{\omega - kv_{yi}}{kv_{\text{th}i}}\right)\right]\right\} = 0, \tag{48}$$

where λ_D^2 equals $T_e/(4\pi Ne^2)$ and N is the plasma density. Equation (47) is the governing equation. We would like to solve it with the boundary conditions

$$\delta\Phi(0) = 0 = \delta\Phi(L) , \tag{49}$$

where the anode is located at $x = 0$ and the cathode at $x = L$. Equation (47) with the boundary conditions is an eigenvalue equation for the eigenfunction $\delta\Phi$ and the eigenvalue ω.

III. CALCULATION OF THE EIGENFUNCTIONS AND THE GROWTH RATES OF THE INSTABILITY

We rewrite the governing equation (47) as follows:

$$\frac{d^2\delta\Phi}{d\bar{x}^2} - [(kd)^2 - g\exp(-\bar{x})]\delta\Phi = 0 . \tag{50}$$

The new eigenvalue g is

$$g = -\left(\frac{d}{\lambda_{DA}}\right)^2\left\{1 - I_0(\zeta)\exp(-\zeta) + \frac{kv_{De}}{(\omega - kv_E)} I_0(\zeta)\exp(-\zeta) + \left(\frac{T_e}{T_i}\right)\left[1 + \left(\frac{\omega - kv_{yi}}{kv_{\text{th}i}}\right)Z\left(\frac{\omega - kv_{yi}}{kv_{\text{th}i}}\right)\right]\right\}, \tag{51a}$$

where

$$\bar{x} \equiv x/d . \tag{51b}$$

It is readily seen that Eq. (50) [with the boundary conditions (49)] is Sturmian and that g takes an infinite number of real and positive values g_n that accumulate at infinity.[21] This series of eigenvalues g_n depends on the two parameters kd and L/d. Once g_n is found for given kd and L/d, the eigenvalues ω_n are found by solving Eq. (51), which is equivalent to the local dispersion relation (48) with density,

$$N = N_A(kd)^2/g_n . \tag{52}$$

The density satisfies Eq. (52) at $x = x_{t,n}$, where

$$\exp(-x_{t,n}/d) = (kd)^2/g_n . \tag{53}$$

The point $x_{t,n}$, which is the turning point of the nth mode, is positive if g_n is larger than $(kd)^2$. We note that the only turning point of each mode is also an inflection point. The turning point is the only inflection point that is not necessarily a zero of the function.

We can now interpret the local analysis and the local dispersion relation (48) within the framework of the nonlocal analysis. The eigenvalues g_n are found by solving the nonlocal problem [Eq. (50)]. For each eigenvalue g_n, the single turning point $x_{t,n}$ is found through Eq. (53). The local dispersion relation is valid at this point and the roots ω_n of this local dispersion relation are the eigenvalues corresponding to the mode whose turning point is at $x_{t,n}$. Since the eigenvalues g_n comprise a real increasing series, the turning points also comprise a real increasing series. If the distance between successive g_n is small, the distance between neighboring turning points is small as well. Thus close to each point there is a turning point of some mode, and the local dispersion relation at each point is approximately satisfied by the eigenvalues of the mode.

The stability of the modes is determined by the roots of the local dispersion relation at the turning points of the modes. Since the only nonuniform plasma parameter is the density, the difference in stability between the various modes results from the dependence of the roots of the local dispersion relation on the plasma density.

Before we refer to the extensive information accumulated about the roots of the local dispersion relation in order to draw conclusions about the stability of the nonlocal modes, we discuss the explicit solutions of the nonlocal equation (50) and the profiles of the eigenfunctions.

The explicit solution of our governing equation (50) is

$$\delta\Phi(x) = AJ_{2kd}(2g^{1/2}e^{-x/2d}) + BY_{2kd}(2g^{1/2}e^{-x/2d}) . \tag{54}$$

From the boundary conditions (49) we obtain

$$AJ_{2kd}(2g^{1/2}) + BY_{2kd}(2g^{1/2}) = 0 \tag{55a}$$

and

$$AJ_{2kd}(2g^{1/2}e^{-L/2d}) + BY_{2kd}(2g^{1/2}e^{-L/2d}) = 0 . \tag{55b}$$

Equations (55) have a nontrivial solution for A and B if the following dispersion relation is satisfied:

$$J_{2kd}(2g^{1/2})Y_{2kd}(2g^{1/2}e^{-L/2d}) - J_{2kd}(2g^{1/2}e^{-L/2d})Y_{2kd}(2g^{1/2}) = 0 . \tag{56}$$

Let us assume that L/d is very large. Since we want $\delta\Phi$ to be bounded also at infinity we require that B in Eq. (54) be zero. The electrostatic potential is therefore

$$\delta\Phi = J_{2kd}(2g^{1/2}e^{-x/2d}), \quad L \to \infty . \tag{57}$$

The requirement that $\delta\Phi$ is zero at the anode yields the dispersion relation

$$J_{2kd}(2g^{1/2}) = 0 . \tag{58}$$

The eigenvalues g_n satisfy

$$g_n = R_n^2/4 , \tag{59}$$

where R_n are the zeros of the Bessel function J_{2kd}. Note that all these zeros are real. The eigenfunctions are now written as

$$\delta\Phi_n(x) = J_{2kd}(R_n e^{-x/2d}) . \tag{60}$$

Here R_m are the positive zeros only. We do not refer to the negative zeros $-R_n$ because their substitution into the expressions (60) does not add independent solutions to Eq. (50). Using Eq. (59) we write Eq. (53) as

$$\exp(-x_{t,n}/2d) = 2kd/R_n . \tag{61}$$

Since the zeros R_n are larger than the order $2kd$,[22] there exists a positive turning point $x_{t,n}$ for each mode. Let us denote by $x_n^{(m)}$ the mth zero of the eigenfunction $\delta\Phi_n$. There are $n+1$ such zeros in the interval $[0, \infty)$. These zeros satisfy

$$\exp(-x_n^{(m)}/2d) = R_{n-m}/R_n, \quad m = 0,\ldots,n . \tag{62}$$

Since R_m is larger than $2kd$, we find, using (61) and (62), that

$$x_{t,n} > x_n^{(m)}, \quad m = 0,\ldots,n . \tag{63}$$

As expected, all the zeros are located between the anode and the turning point. The value of the eigenfunction at the turning point is

$$\delta\Phi_n(x_{t,n}) = J_{2kd}(2kd) \tag{64}$$

following (62), and is of the same form for all the modes.

Since it usually is so, we now assume that kd is much larger than unity. In that case the order of the Bessel function is large, and the first zero is approximately[22]

$$R_0 = 2kd\{1 + 1.86(2kd)^{-2/3} + O[(2kd)^{-4/3}]\}, \quad kd \gg 1. \tag{65}$$

The first maximum M_0 of the Bessel function for large order is[22]

$$M_0 = 2kd\{1 + 0.81(2kd)^{-2/3} + O[(2kd)^{-4/3}]\}, \quad kd \gg 1. \tag{66}$$

The locations of the last zero $x_n^{(n)}$, the last maximum $x_n^{(\max)}$, and the turning point of the nth mode are

$$x_n^{(n)} \cong 2d\ln(R_n/R_0), \tag{67a}$$

$$x_n^{(\max)} \cong 2d\ln(R_n/M_0), \tag{67b}$$

$$x_{t,n} = 2d\ln(R_n/2kd). \tag{67c}$$

Using the approximate expressions (65) and (66), we obtain

$$x_n^{(\max)} \cong x_n^{(n)} + 1.05(2kd)^{-2/3}2d, \tag{68a}$$

$$x_{t,n} \cong x_n^{(n)} + 1.86(2kd)^{-2/3}2d. \tag{68b}$$

The turning point is very close to the last maximum of the eigenfunction, where the amplitude is largest. The eigenvalues of each mode are approximately the roots of the local dispersion relation (48) at that maximum or at the last zero. For the fundamental mode the eigenvalues are found by solving the local dispersion relation at the anode.

When kd is large the distance between successive zeros of J_{2kd} is small,

$$(R_{n+1} - R_n)/R_0 \cong (R_{n+1} - R_n)/2kd \ll 1. \tag{69}$$

Following (67c), the distance between the turning points of successive modes is also small,

$$(x_{t,n+1} - x_{t,n})/d \ll 1. \tag{70}$$

Close to each value of x there exists a turning point of some mode, the eigenvalues of which are determined by the plasma parameters at x. That mode also has its largest value near x. In this sense these modes are local modes.

We now compare the growth rates of the instability of the various modes. The various modes differ in the plasma density at the respective turning points. Thus we have to examine the dependence of the instability on the density in the local dispersion relation. For this purpose we refer to the results obtained by previous extensive studies of the local dispersion relation.

We start with the case of small drift velocity where $|v_E| \ll v_{\text{th}i}$, or, following (32),

$$(c/eB_0d)[T_{\perp i} - rT_{\perp e}/(1+r)] \ll v_{\text{th}i}. \tag{71}$$

In that case the real and imaginary parts of ω are[5]

$$\omega_r = kv_E\left[1 - I_0(\zeta)\exp(-\zeta)\left(\frac{v_{De}}{v_E}\right) \times\left((k\lambda_D)^2 + \frac{T_e}{T_i} + 1 - I_0(\zeta)\exp(-\zeta)\right)^{-1}\right], \tag{72a}$$

$$\omega_i = \sqrt{\pi}\frac{T_e}{T_i}\frac{(\omega_r - kv_E)^2(\omega_r - kv_{yi})}{kv_{\text{th}i}I_0(\zeta)\exp(-\zeta)kv_{De}} = \sqrt{\pi}\frac{T_e}{T_i}\frac{v_{De}}{v_{\text{th}i}}\left((k\lambda_D)^2 + \frac{T_e}{T_i} + \zeta\right)^{-2} \cdot k\left[-v_{Di} - v_{De}\left((k\lambda_D)^2 + \frac{T_e}{T_i} + \zeta\right)^{-1}\right]. \tag{72b}$$

The value of the plasma dispersion function was approximated for a small argument. For notational convenience we write T_j for $T_{\perp j}$.

We now examine the dependence of the growth rate on k and on λ_D. The dependence on λ_D tells us whether the fundamental mode is the most unstable mode (when ω_i decreases with λ_D). Equating the derivative of ω_i with respect to k to zero we find that, for given λ_D, the maximum value of ω_i is obtained for $k = k_M$, where

$$k_M^2 = (T_e/T_i)/(\lambda_D^2 + r_L^2/2). \tag{73}$$

The value of ω_i is then

$$\omega_i = -\frac{\sqrt{\pi}}{8}\left(\frac{T_i}{T_e}\right)^{1/2}\frac{v_{De}v_{Di}}{v_{\text{th}i}(\lambda_D^2 + r_L^2/2)^{1/2}}. \tag{74}$$

On the other hand, equating the derivative of ω_i with respect to λ_D^2 to zero, we find that for given k the maximum value of ω_i is obtained for $\lambda_D = \lambda_{DM}$, where

$$\lambda_{DM}^2 + \frac{r_L^2}{2} = \frac{1}{2}\left(\frac{T_e}{T_i}\right)\frac{1}{k^2}. \tag{75}$$

The value of ω_i is then

$$\omega_i = -\frac{4\sqrt{\pi}}{27}\frac{T_i}{T_e}\frac{v_{De}v_{Di}}{v_{\text{th}i}}k. \tag{76}$$

For a given k, if λ_D^2 is larger (smaller) than λ_{DM}^2, ω_i decreases (increases) with increasing (decreasing) λ_D^2. Therefore if λ_{DM}^2 is smaller than λ_{DA}^2, the fundamental mode is the most unstable. If λ_{DM}^2 is larger than λ_{DA}^2, some higher mode is more unstable. We write this condition in an equivalent way. For

$$k^2 < k_c^2 \equiv \frac{1}{2}\frac{T_e}{T_i}\frac{1}{(\lambda_{DA}^2 + r_L^2/2)}, \tag{77}$$

the maximum growth rate ω_i is obtained for a mode of λ_D^2, which satisfies

$$\lambda_D^2 + \frac{r_L^2}{2} = \frac{1}{2}\frac{T_e}{T_i}\frac{1}{k^2}. \tag{78}$$

The eigenvalue g_n of that mode is

$$g_n = \frac{\lambda_D^2}{\lambda_{DA}^2}k^2d^2 = \left(\frac{1}{2}\frac{T_e}{T_i} - k^2\frac{r_L^2}{2}\right)\frac{d^2}{\lambda_{DA}^2}, \tag{79}$$

following (52) and (78).

In summary, the maximum growth rate is

$$\omega_i = \begin{cases} \sqrt{\pi}\,\dfrac{T_e}{T_i}\dfrac{v_{De}}{v_{\mathrm{th}i}}\left((k\lambda_{\mathrm{D}A})^2+\dfrac{T_e}{T_i}+\zeta\right)^2 k\left[-v_{Di}-v_{De}\left((k\lambda_{\mathrm{D}A})^2+\dfrac{T_e}{T_i}+\zeta\right)^{-1}\right], & k^2 \geqslant \dfrac{1}{2}\dfrac{T_e}{T_i}\dfrac{1}{(\lambda_{\mathrm{D}A}^2+r_{\mathrm{L}}^2/2)}, \\ -\dfrac{4\sqrt{\pi}}{27}\dfrac{T_i}{T_e}\dfrac{v_{De}v_{Di}}{v_{\mathrm{th}i}}k, & k^2 < \dfrac{1}{2}\dfrac{T_e}{T_i}\dfrac{1}{(\lambda_{\mathrm{D}A}^2+r_{\mathrm{L}}^2/2)}. \end{cases} \tag{80}$$

Note that

$$k_c^2/k_{\max}^2 = \tfrac{1}{2} \tag{81}$$

and that

$$\frac{\omega_i(k=k_c)}{\omega_i(k=k_{\max})} = \frac{4}{27}k\left(\frac{T_i}{T_e}\right)^{1/2} 8(\lambda_{\mathrm{D}A}^2 + r_{\mathrm{L}}^2/2)^{1/2} = \left(\frac{32}{27}\right)\frac{1}{2}. \tag{82}$$

This ratio is close to 1.

IV. NUMERICAL EXAMPLES AND DISCUSSION

For the numerical examples we examine a plasma of parameters equal to those of the experiment at the Weizmann Institute.[1] The electron and ion temperatures are 7 and 25 eV, respectively. The intensity of the external uniform magnetic field is 7.2 kG and the plasma density at the anode is 2.2×10^{15} cm^{-3}. The plasma thickness is about 0.1 cm. The electron Debye length at the anode $\lambda_{\mathrm{D}A}$ is 4×10^{-5} cm while the electron Larmor radius r_{L} is 10^{-3} cm. For such small values of $2\lambda_{\mathrm{D}A}^2/r_{\mathrm{L}}^2$ (5×10^{-3}) the electron Debye length λ_{D} could be neglected in the local dispersion relation relative to the electron Larmor radius and therefore the growth rate of the instability is independent of λ_{D} (and of the density). This is usually the case in plasmas of not too low a density. Note that the ratio $\lambda_{\mathrm{D}}^2/r_{\mathrm{L}}^2$ is independent of the temperature. The various modes differ only in the plasma density at their turning points, and, since this density does not affect the eigenvalues, all the modes grow at the same rate. The fluctuating fields resulting from the instability are expected to fill the plasma uniformly.

Figure 2 shows the growth rate of the instability ω_i (calculated from the local dispersion relation) versus the wavenumber k for the parameters of the Weizmann experiment. Figure 3 shows the eigenmode profiles $\delta\Phi_n$ for $n = 0, 9, 20$. The parameter kd equals 60 and corresponds to the maximum growth rate in the Weizmann experiment.

Figure 4 shows the growth rate of the instability versus the density for a plasma of much lower density. The temperatures, the magnetic field, and the slab thickness are as in

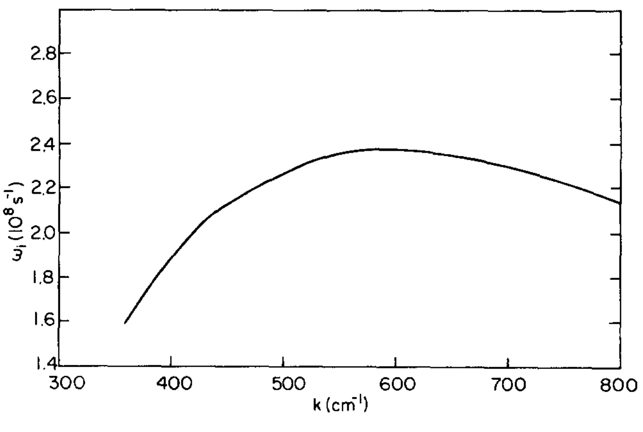

FIG. 2. The growth rate ω_i versus the wavenumber k. The parameters are $T_e = 7$ eV, $T_i = 25$ eV, $d = 0.1$ cm, $B_0 = 7.2$ kG, and $N_A = 2.2\times10^{15}$ cm^{-3}.

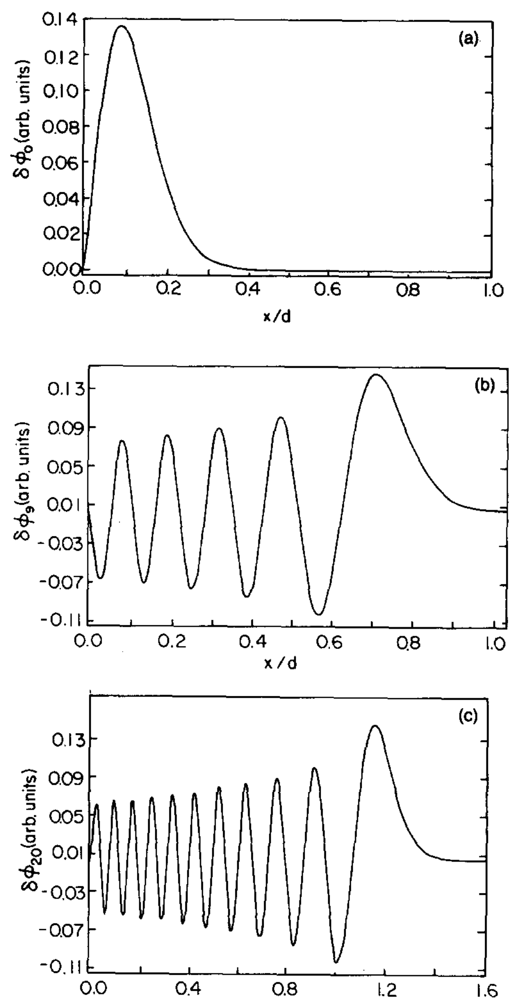

FIG. 3. The profiles of the eigenmodes $\delta\Phi_n$ for $kd = 60$. (a) $n = 0$, (b) $n = 9$, (c) $n = 20$.

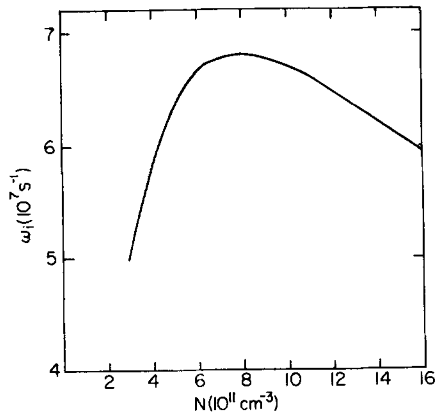

FIG. 4. The growth rate ω_i versus the density N. The parameters are $T_e = 7$ eV, $T_i = 25$ eV, $d = 0.1$ cm, $B_0 = 7.2$ kG, $k = 162$ cm^{-1}.

the previous example. For such a plasma of about 10^{12} cm^{-3}, the growth rate of the instability is sensitive to the density. In Fig. 4 the wavenumber k equals 162 cm^{-1} and is smaller than k_c. As shown in the figure the dependence of ω_i on the density N is not monotonic and has a maximum as given in Eq. (76). In such a plasma the various modes have different growth rates. If the density at the anode N_A is 1.5×10^{12} cm^{-3}, the most unstable mode among the modes of $k = 162$ cm^{-1} is that mode whose turning point is located where the density falls to about 8×10^{11} cm^{-3}. Higher and lower modes will grow at a smaller rate. Note, however, that the most unstable mode is that of a wavenumber k equal to k_M, which, following (73), is about 300 cm^{-1}. The maximal growth rate is of the fundamental mode and, following (74), is about 10^{-8} sec^{-1}.

In formulating our model problem we made some simplifying assumptions. We now discuss the implications of these assumptions and the possibility of generalization to a more realizable model.

In the choice of the equilibrium there was some arbitrariness. Even among the time-independent solutions of the Vlasov equation we could have chosen distribution functions other than (20). In the particular form of the distribution function we analyzed, all the quantities are uniform, except for the density. It would be interesting to study the stability of equilibria that are nonuniform in other parameters as well, especially since the rate of instability depends only weakly on the density. Such nonuniform equilibria could possess very localized modes as the most unstable modes. The level of fluctuations would be nonuniform and may result in nonuniform diffusion coefficients in the description of the anomalous diffusion of the plasma.

In the Weizmann experiment the electron–ion collision frequency is of the same order of magnitude as the frequency of the instability. Collisions may thus have a stabilizing effect in this case.

We note that the equilibrium distribution functions we assumed are not valid near the walls. Since the cathode is far away, the main inconsistency is near the anode. For each species the distribution function is probably incorrect within a distance smaller than the Larmor radius. For the electrons this distance is very small and would probably not affect much. For the ions this distance is larger. The assumption that the ions are unmagnetized seems to be a good one, but it is not clear whether their distribution function is Maxwellian near the wall.

We have noted that there are cases of interest in which the high modes are more unstable than the fundamental mode. Our analysis is not valid for very high modes when the wave fields vary in the x direction on a scale length comparable to the electron Larmor radius. The analysis should be modified to include those FLR effects in the x direction. We anticipate, however, that the incorporation of such FLR effects into our model will not change the results substantially.

It is also possible that the spatial dependence of the wave at saturation will differ from the spatial dependence in the exponential growth regime. Thus it is hard to conclude from our linear analysis anything definite about the spatial dependence of the coefficients of the anomalous diffusion.

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