Electron beam equilibria with self-fields for a free electron laser with a planar wiggler

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A general formalism for non-neutral cold relativistic planar steady flows is developed and applied to the study of the equilibrium of a sheet electron beam in a planar wiggler free electron laser. The full transverse dependence of the wiggler field as well as the equilibrium self-fields of the beam are included. In particular, the betatron oscillations in the presence of self-fields are studied. For a thick beam equilibrium with a particular density profile it is shown that the betatron oscillations are eliminated. For a thin beam configuration the paraxial approximation is employed and it is also shown that for some critical density there are no betatron oscillations. If the density is larger than this critical density the beam oscillates with the betatron frequency but there are no trajectory crossings and the beam preserves its cold fluid nature. The single-particle equations of motion are also considered in the presence of both planar wiggler and planar self-fields. It is shown that in some cases the particles oscillate with a reduced betatron frequency, in contrast to the previous case of cold fluid motion where the self-fields do not change the betatron frequency. For the study of the betatron oscillations in the thick beam equilibrium a two-space scale method is employed. For the thin beam within the paraxial approximation the Floquet theory for equations with periodic coefficients is used.

I. INTRODUCTION

Free electron lasers (FEL's) are a subject of intensive theoretical and experimental study.[1] In some experiments a planar wiggler is employed to force the electron beam into the wiggling orbit necessary for the FEL interaction.[2] Theoretical treatments of FEL's that employ planar wigglers sometimes neglect the transverse dependence of the wigglers.[3–5] Such an approximation fails for long interaction length FEL's because of the betatron oscillations. These oscillations, with a wavelength much longer than the wiggler wavelength, may introduce thermal spread into the beam, cause the beam envelope to oscillate, and change the beam density substantially over a betatron wavelength. One approach in addressing the betatron oscillations is to assume that the beam momentum spread and the beam initial radius are such that the beam envelope does not vary.[6] In that case the equilibrium density of the beam is assumed to be constant. The betatron oscillations and the transverse gradients are then modeled one-dimensionally with the introduction of an effective temperature of the beam. A second approach is to assume that the beam envelope oscillates with some small amplitude at the betatron oscillation wavelength.[7] That picture of the beam does not describe explicitly the equilibrium and the betatron oscillation is expressed by means of the varying envelope.

It is difficult to compress a high current beam into a small cross section. For a high quality finite cross-section beam, the thermal spread of the beam could be insignificant in determining the beam dynamics. Instead, the equilibrium self-fields of the beam could play a major role. To explore the validity of that view, we emphasize in this paper a cold fluid model, which incorporates the equilibrium self-fields of the beam. The solutions of the cold fluid equations may have singularities coming from the crossings of streamlines, equivalent to the well-known crossings of single-particle trajectories. At these singular points the cold fluid model breaks down. We explore regimes where the cold fluid model admits regular solutions. Such solutions describe equilibria with no trajectory crossings, and they may be important for FEL applications. The singularities that could be present have been eliminated by the presence of the self-fields, which act to oppose the restoring force of the wiggler field.

Since for some cases crossing of streamlines occurs, we also examine the single-particle equations of motion in the presence of given self-fields. We derive the constants of the motion and demonstrate some of the effects of the self-fields. This formulation of the single-particle equations of motion could be the basis for a kinetic model of equilibria when crossings of trajectories occur.

Because the wiggler is planar we study an electron beam of the same structure, a rectangular or a sheet electron beam. Parenthetically, one should bear in mind that, because of the betatron oscillations, an electron beam, which is initially cylindrical, takes an elongated form as it propagates; a sheet beam, on the other hand, preserves its rectangular shape even though its thickness may vary. The electron beam is assumed to propagate in the z direction and to be infinite in extent in the x direction, the direction perpendicular to both the wiggler field and the transverse gradient of the wiggler field. This idealization is possible practically because the planar wiggler has a weak x dependence. The experimental use of wide sheet beams enables one to increase the current by increasing the beam cross section in the x direction. The model problem we study is finally a two-dimensional problem in which all the quantities are independent of x and depend on y and z only.

We start by formulating in Sec. II the general cold fluid equations for steady planar relativistic non-neutral flows. Similarly to our previous study of helically symmetric flows,[8] we replace the equations for the fields and for the fluid moments by equations for three scalar functions: a magnetic flux function, a fluid stream function, and an electrostatic potential. The reduction in the number of unknown functions simplifies the equations, clarifies what boundary conditions have to be specified, and makes it easier to derive approximate solutions by asymptotic expansions. The equations allow general two-dimensional electric and magnetic fields and include the full effects of the steady self-fields of the beam. Also in Sec. II we describe the single-particle motion in static electric and magnetic fields independent of x by writing Hamilton's equations for the system.

The general formalism of Sec. II is applied in Sec. III for a situation common in FEL's, in which the beam is thin in the y direction. We apply the paraxial approximation to both the cold fluid system and to the single-particle motion. The cold fluid equations are reduced to an inhomogeneous second-order, linear ordinary differential equation in z for the reciprocal of the density. The coefficients in the equation are given in terms of the applied axial electric and magnetic fields. The inhomogeneous term in the equation comes from the self-fields of the beam. When we assume an external, periodic, planar wiggler, the equation has periodic coefficients. Applying the Floquet theory we look for periodic solutions with the wiggler periodicity and with no betatron motion, and for almost periodic solutions with the additional periodicity of the betatron motion. We also examine the conditions under which the solution has zeros, which correspond to crossings of trajectories and to the breakdown of the fluid model. We show that there is a critical density for which the betatron oscillations are eliminated, and the solution is periodic with neither secular nor almost periodic terms. If the density is higher than this critical density the beam then oscillates with the wiggler periodicity and also with the betatron periodicity around a new equilibrium plane. In this case there are no crossings of trajectories and the cold fluid picture remains valid. When the density is lower, crossings occur and the cold fluid model fails. By applying the paraxial approximation to the single-particle equations of motion we derive a homogeneous second-order equation. We write the constants of the motion in the combined wiggler field and self-fields. These constants of motion could become the basis of a kinetic model. We show that the periodicity of the motion of a particle is modified in the presence of z-independent self-fields. We note the remarkable difference between this motion with a reduced betatron frequency and the cold fluid solution in the presence of self-fields for which the betatron frequency is not modified. These are clearly two different equilibria, which will probably result in different FEL interactions.

In Sec. IV we return to the general equations with the full transverse dependence. Rather than employing the paraxial approximation we expand the equations in a small parameter that measures the ratio of the perpendicular to parallel momenta. We use a two-space-scale method and show that for a beam of a given energy propagating along a planar wiggler there are density profiles for which the self-fields balance the wiggler focusing and the betatron oscillations are eliminated. For densities near that critical density profile we show that betatron oscillations are present, that their wavelength varies across the profile, that no crossings occur, and that the cold fluid model remains valid.

II. THE COLD FLUID PLANAR STEADY FLOW MODEL

We consider the steady flow of zero-temperature electrons in which all quantities depend on the y and z rectangular coordinates only. We take the flow to be essentially in the z direction, with y as the coordinate of spatial extent, and x to be an ignorable coordinate. With appropriate nondimensionalization of the reduced momentum $\mathbf{u}$, the density n, the electric field $\mathbf{E}$, and the magnetic field $\mathbf{B}$, the cold fluid model reduces to conservation of mass

$$\nabla\cdot(n\mathbf{u}) = 0, \tag{1}$$

conservation of momentum

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = -\gamma\mathbf{E} - \mathbf{u}\times\mathbf{B}, \tag{2}$$

where

$$\gamma^2 = 1 + \mathbf{u}^2, \tag{3}$$

and Maxwell's equations

$$\nabla\times\mathbf{E} = 0, \tag{4}$$

$$\nabla\cdot\mathbf{E} = -n\gamma, \tag{5}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{6}$$

$$\nabla\times\mathbf{B} = -n\mathbf{u}. \tag{7}$$

For steady flows that are independent of x we may introduce a magnetic flux function $\psi(y,z)$ such that

$$B_y = -\psi_{,z}, \tag{8a}$$

$$B_z = \psi_{,y}, \tag{8b}$$

and an electrostatic potential $\phi(y,z)$ such that

$$\mathbf{E} = -\nabla\phi. \tag{9}$$

It is also possible to introduce a streamfunction $\chi(y,z)$ such that

$$nu_y = -\chi_{,z}, \tag{10}$$

$$nu_z = \chi_{,y}, \tag{11}$$

and (1), (4), and (6) are satisfied identically. It then follows from (7) that

$$B_x = \chi(y,z), \tag{12}$$

and

$$\Delta\psi = -nu_x, \tag{13}$$

while Poisson's equation (5) has the usual form

$$\Delta\phi = n\gamma. \tag{14}$$

With the definitions and equations [(8)–(14)], all the equations of the system (1)–(7) are satisfied except (2) and the definition (3).

When we dot **u** into (2) we obtain the usual form of conservation of energy,

$$(\mathbf{u}\cdot\nabla)(\gamma-\phi)=0,$$

so that we may introduce an arbitrary function of χ, $E(\chi)$, such that

$$\gamma = E(\chi)+\phi(y,z). \tag{15}$$

In addition, the x component of (2) yields

$$(\mathbf{u}\cdot\nabla)(u_x+\psi)=0,$$

so that we may introduce another function $F(\chi)$ such that

$$u_x = F(\chi)-\phi. \tag{16}$$

When we combine (10), (11), (15), and (16) we obtain an explicit expression for n,

$$n(y,z) = |\nabla\chi|/\{[E(\chi)+\phi]^2-1 - [\psi-F(\chi)]^2\}^{1/2}. \tag{17}$$

The remaining component of (2) can be cast in the form

$$\begin{aligned} L\chi &\equiv \chi_{,y}^2\chi_{,zz}-2\chi_{,y}\chi_{,z}\chi_{,yz}+\chi_{,z}^2\chi_{,yy} \\ &= -n[n(E(\chi)+\phi)\nabla\phi\cdot\nabla\chi \\ &\quad - n(\psi-F(\chi))\nabla\psi\cdot\nabla\chi-\chi\nabla\chi\cdot\nabla\chi]. \end{aligned} \tag{18}$$

Maxwell's equations assume the final form

$$\Delta\phi = n(E(\chi)+\phi) \tag{19}$$

and

$$\Delta\psi = n(\psi-F(\chi)). \tag{20}$$

Thus cold steady planar flow is characterized by the system (17)–(20). We note in passing that this system was contained in our earlier paper[8] on helically symmetric flows following a simplification and transformation. If one sets the helical wavenumber k to zero, expresses all quantities in rectangular rather than cylindrical coordinates, and then makes a cyclic permutation of the independent variables $(x,y,z)\rightarrow(y,z,x)$, one obtains the above results. It seems, however, more natural to rederive them directly, rather than rely on the simplification.

Since we intend to make some connection with single-particle motion, it is useful to describe the associated problem of single-particle motion in static electric and magnetic fields independent of x. It is easy to see that an appropriate vector potential **A** for the magnetic field is

$$\mathbf{A} = (-\psi,0,A_z),$$

where

$$A_{z,y}=\chi(y,z)=B_x(y,z),$$

so that the Hamiltonian for the motion of an electron is

$$H=[1+(p_x-\psi)^2+p_y^2+(p_z+A_z)^2]^{1/2}-\phi. \tag{21}$$

Since the x coordinate is ignorable, p_x is constant and $u_x = p_x-\psi$ [compare (16)]. When we take p_x as constant, we observe that (21) gives the Hamiltonian for the two-dimensional motion in the y-z plane and that the Hamiltonian is time independent. Thus the Hamiltonian is constant, or $\gamma = H+\phi$ [compare (15)]. Now, Hamilton's equations are four, first-order ordinary differential equations. Since the Hamiltonian is time independent, we may drop the order of the system from fourth order to second order by a simple transformation in which we solve (23) for p_z as a function of the other variables,

$$p_z = [(H+\phi)^2-1-(p_x-\psi)^2-p_y^2]^{1/2}-A_z, \tag{22}$$

and we take $-p_z$ as the generalized Hamitonian with conjugate "time" z. This transformation is clearly canonical and the form of Hamilton's equations is[9]

$$H = \text{const},$$

$$\frac{dp_y}{dz} = -A_{z,y}+\frac{(H+\phi)\phi_{,y}-(\psi-p_x)\psi_{,y}}{[(H+\phi)^2-1-(p_x-\psi)^2-p_y^2]^{1/2}}, \tag{23a}$$

$$\frac{dy}{dz}=\frac{p_y}{[(H+\phi)^2-1-(p_x-\psi)^2-p_y^2]^{1/2}}, \tag{23b}$$

$$\frac{dp_z}{dz} = -A_{z,z}+\frac{(H+\phi)\phi_{,z}-(\psi-p_x)\psi_{,z}}{[(H+\phi)^2-1-(p_x-\psi)^2-p_y^2]^{1/2}}, \tag{24a}$$

$$\frac{dt}{dz}=\frac{H+\phi}{[(H+\phi)^2-1-(p_x-\psi)^2-p_y^2]^{1/2}}. \tag{24b}$$

The system (22)–(24) is, of course, equivalent to the usual treatments of single-particle motion, but it simplifies and clarifies some of the approximations commonly made. We compare the properties of the cold fluid and single-particle motion when it is possible.

III. THE PARAXIAL APPROXIMATION

We apply the paraxial approximation to our cold fluid system and we compare the results with what we can obtain for single-particle motion. We apply the Floquet theory of ordinary differential equations with periodic coefficients. When we make the usual approximations of a small applied magnetic wiggler and no applied electric field we obtain a series of explicit results on the appearance or nonappearance of betatron oscillations and orbit crossings, which imply the breakdown of the cold fluid model.

We return to our cold fluid system (17)–(20), and we wish to expand the solution in a power series in y, where we assume that y is small. In particular, we assume

$$\begin{aligned} \chi &= ya_1(z)+y^3a_3(z)+\cdots, \\ \psi &= \psi_0(z)+(y^2/2)\psi_2(z)+\cdots, \\ \phi &= \phi_0(z)+(y^2/2)\phi_2(z)+\cdots, \\ E(\chi) &= E_0+E_2\chi^2/2+\cdots, \\ F(\chi) &= F_2\chi^2/2+\cdots. \end{aligned} \tag{25}$$

In (25), we have set $F_0=0$. We could easily assume $F_0\neq 0$, or equivalently we could add this constant to $\psi_0(z)$ without changing the system. Equivalently, we could also take E_0 to be zero by adding a constant to $\phi_0(z)$, but since E_0 corresponds to the particle energy, it is physically desirable to make this constant explicit. On the other hand, F_0 essentially corresponds to the mean x component of momentum, and we expect it to be small and not physically significant. In any case a Lorentz transformation in the x direction can modify the mean x component of momentum as desired without substantially affecting an x-independent solution.

When we insert (25) into (17), (19), and (20) we find to lowest order

$$n = n(z) = a_1(z)/\{[E_0 + \phi_0(z)]^2 - 1 - [\psi_0(z)]^2\}^{1/2} + \cdots, \quad (26)$$

$$\phi_2(z) = -\phi_0''(z) + n(z)[E_0 + \phi_0(z)], \quad (27)$$

$$\psi_2(z) = -\psi_0''(z) + n(z)\psi_0(z), \quad (28)$$

while (18) becomes

$$(a_1)^2 a_1'' - 2a_1(a_1')^2 = -a_1^2 a_1^1 \Delta'/\Delta - a_1^4/\Delta^3 + a_1^3[(E_0 + \phi_0)\phi_0'' - \psi_0\psi_0'']/\Delta^2, \quad (29)$$

where

$$\Delta^2(z) = [E_0 + \phi_0(z)]^2 - 1 - [\psi_0(z)]^2. \quad (30)$$

We define

$$b_1(z) = 1/a_1(z), \quad (31)$$

and (29) becomes, after division by $[a_1(z)]^4/\Delta^2$,

$$\Delta(\Delta b_1')' + \{[E_0 + \phi_0(z)]\phi_0''(z) - \psi_0(z)\psi_0''(z)\}b_1(z) = 1/\Delta(z). \quad (32)$$

Thus the flow is given in the paraxial approximation in terms of the solution of the linear, inhomogeneous, differential equation for $b_1(z)$, when the electric potential and magnetic streamfunction are given on axis. We see from the paraxial approximation for χ that the equation of a streamline, or particle trajectory, is

$$y = \chi_0 b_1(z), \quad (33)$$

where χ_0 is a constant. Thus particle trajectories cross, and the cold fluid approximation fails, when $b_1(z)$ vanishes. No other trajectory crossings occur.

If we assume that $\phi_0(z)$ and $\psi_0(z)$ are given periodic functions of z of period $L = 2\pi/k$, then (32) for $b_1(z)$ is of typical Floquet form.[10] In general, we expect there to exist a unique solution of the equation of period L. The general solution of the equation is the sum of this solution plus a solution of the homogeneous equation. The solutions of the homogeneous equation can, in principle, be unbounded in z, or they may be almost periodic functions corresponding to the appearance of betatron oscillations. We cannot, in general, decide whether or not $b_1(z)$ vanishes for any values of z.

We now apply the explicit Floquet theory to a particular case of (32) that is of some general interest. We assume that there are no applied electrostatic fields, so that $\phi_0(z) \equiv 0$, that the applied wiggler has the simple form $\psi_0 = B_w \cos z$, and that

$$2\epsilon^2 \equiv B_w^2/(E_0^2 - 1) \quad (34)$$

is small. We define

$$\delta^2(z) = \Delta^2/(E_0^2 - 1) = 1 - \epsilon^2(1 + \cos 2z), \quad (35)$$

and (32) becomes

$$\delta(\delta b_1')' + \epsilon^2(1 + \cos 2z)b_1 = (E_0^2 - 1)^{-3/2}/\delta. \quad (36)$$

If δ were identically constant then (36) would be an inhomogeneous Mathieu equation. The following results are fairly standard, but we present them with succinct derivations and reinterpret in the context of our problem.

We consider the homogeneous equation

$$\delta(z)[\delta(z)y']' + \epsilon^2(1 + \cos 2z)y = 0. \quad (37)$$

For any two solutions $y_1(z)$ and $y_2(z)$,

$$\delta(z)[y_1(z)y_2'(z) - y_1'(z)y_2(z)] = \text{const}, \quad (38)$$

so that

$$W(z + n\pi) = W(z), \quad (39)$$

where n is an integer and $W(z)$ is the Wronskian of the two solutions or

$$W(z) = y_1(z)y_1'(z) - y_2'(z)y_2(z). \quad (40)$$

We construct two fundamental solutions of (37) by the imposition of the boundary conditions $y_1(0) = y_2'(0) = 1$ and $y_1'(0) = y_2(0) = 0$. An elementary calculation shows that

$$y_1(z) = 1 - \epsilon^2\left(\frac{z^2}{2} + \frac{1 - \cos 2z}{4}\right) + O(\epsilon^4) \quad (41)$$

and

$$y_2(z) = z - \epsilon^2\left(\frac{z}{4} + \frac{z^3}{6} - \frac{z\cos 2z}{4}\right) + O(\epsilon^4), \quad (42)$$

and these solutions are uniformly valid in $0 \leqslant z \leqslant \pi$. Further, a particular solution of (36) valid in the same interval is

$$\beta(z) = (E_0^2 - 1)^{-3/2}\{\tfrac{1}{2}z^2 + \epsilon^2[\tfrac{3}{4}z^2 - z^4/24 + \tfrac{1}{8}z^2\cos 2z + \tfrac{5}{16}(1 - \cos 2z)] + O(\epsilon^4)\}. \quad (43)$$

We can now determine whether Eq. (36) has any solutions periodic of period π. The general solution of the equation is

$$b_1(z) = \beta(z) + \lambda y_1(z) + \mu y_2(z), \quad (44)$$

where λ and μ are arbitrary constants. The conditions for a solution period of π are $y(0) = y(\pi)$ and $y'(0) = y'(\pi)$, or

$$\lambda = \lambda\left(1 - \epsilon^2\frac{\pi^2}{2}\right) + \mu\left(\pi - \epsilon^2\frac{\pi^3}{6}\right) + (E_0^2 - 1)^{-3/2}\left(\frac{\pi^2}{2} + O(\epsilon^2)\right), \quad (45a)$$

$$\mu = -\lambda\epsilon^2\pi + \mu[1 - \epsilon^2(\pi^2/2) + (E_0^2 - 1)^{-3/2}[\pi + O(\epsilon^2)]. \quad (45b)$$

It is easy to see that a unique solution exists, and, in particular,

$$\lambda = (E_0^2 - 1)^{-3/2}[\epsilon^{-2} + O(1)], \quad (46a)$$

$$\mu = (E_0^2 - 1)^{-3/2}O(1). \quad (46b)$$

Since $y_1(z) = 1 + O(\epsilon^2)$ on the interval $0 \leqslant z \leqslant \pi$ and $y(z)$ is periodic of period π, we conclude from (41)–(46) that $y(z)$ also has no zero crossings and that the cold plasma model is uniformly valid for the periodic solution. We remind the reader that the general Floquet theory shows that although we calculate results by perturbation expansion, they are valid for all z. For this solution the density for which a periodic solution exists is given by

$$n(z) = \epsilon^2(E_0^2 - 1)[1 + O(\epsilon^2)] = (B_w^2/2)[1 + O(\epsilon^2)]. \quad (47)$$

In order to consider other densities we must return to (36) and obtain the general solution.

The general solution of (36) is the solution $b_1(z)$ given by (44) with λ and μ specified by (45) and (46) plus an arbitrary solution of the homogeneous equation (37). To examine these homogeneous solutions we return to the Floquet theory. The general solution of (37) is of the form

$$w(z) = \tilde{\lambda} y_1(z) + \tilde{\mu} y_2(z), \quad (48)$$

with $\tilde{\lambda}$ and $\tilde{\mu}$ now arbitrary. Although there may not be solutions of period π, we can look for solutions with Floquet exponent σ such that

$$w(z+\pi) = \sigma w(z), \quad (49a)$$

$$w'(z+\pi) = \sigma w'(z). \quad (49b)$$

If (49) holds then

$$w(z+n\pi) = \sigma^n w(z), \quad (50a)$$

$$w'(z+n\pi) = \sigma^n w'(z). \quad (50b)$$

Thus the solutions are bounded if $|\sigma| = 1$ and tend to zero or infinity if $|\sigma| \neq 1$. We may calculate the exponents from the nature of the solutions on the interval $0 \leqslant z \leqslant \pi$ only. The exponents satisfy

$$w(\pi) = \sigma w(0),$$

$$w'(\pi) = \sigma w'(0),$$

or

$$\tilde{\lambda}[y_1(\pi) - \sigma] + \tilde{\mu} y_2(\pi) = 0, \quad (51a)$$

$$\tilde{\lambda} y_1'(\pi) + \tilde{\mu}[y_2'(\pi) - \sigma] = 0, \quad (51b)$$

and in view of (39) and (40) we find easily that

$$\sigma^2 - \sigma[y_1(\pi) + y_2'(\pi)] + 1 = 0. \quad (52)$$

Thus we see that if σ is a root so are $1/\sigma$, σ^*, and $1/\sigma^*$. If $|\sigma| = 1$, then the two roots are σ and σ^*, while if $|\sigma| > 1$, then σ is real and the two roots are σ and $1/\sigma$. For our problem, (52) becomes

$$\sigma^2 - \sigma[2 - \epsilon^2\pi^2 + O(\epsilon^4)] + 1 = 0,$$

or

$$\sigma = 1 \pm i\epsilon\pi + O(\epsilon^2).$$

Since σ is not real we conclude $|\sigma| = 1$, or $\sigma = 1 \pm i\epsilon\pi - \epsilon^2\pi^2/2 + O(\epsilon^3)$ and

$$\sigma = \exp\{\pm i\epsilon\pi[1 + O(\epsilon^2)]\}. \quad (53)$$

We could calculate σ to higher order in ϵ, but (53) suffices. All we must know is that there is an exact Floquet exponent σ of which the right-hand side of (53) is an approximation.

We can now return to (51) and calculate the structure of the solutions. We see that to leading order the solution with Floquet exponent (53) is

$$y_\pm(z) = y_1(z) \pm i\epsilon[1 + O(\epsilon^2)]y_2(z), \quad (54)$$

so that

$$y_\pm(z+n\pi) = \exp\{\pm i\epsilon n\pi[1 + O(\epsilon^2)]\} y_\pm(z). \quad (55)$$

We may obtain useful information from (55) even if $n\epsilon$ is $O(1)$ or large, provided only that $n\epsilon^3$ is small. Thus we could easily take $n \sim 1/\epsilon^2$, so that $n\epsilon \sim 1/\epsilon$ but $n\epsilon^3 \sim \epsilon$. The form (55) shows that for $n\epsilon$ near an even integer, $y_\pm(z+n\pi)$ is approximately $y_\pm(z)$. If this period is incommensurable with π, then this oscillation—the betatron oscillation—is an almost periodic oscillation of period $2/\epsilon$. We can extract still more information from (55). The general solution of the homogeneous equation is clearly

$$y_H(z) = \mathrm{Re}(\{y_1(z) + i\epsilon[1 + O(\epsilon^2)]y_2(z)\}e^{i\omega}), \quad (56)$$

where ω may be taken real, and

$$y_H(z+n\pi) = \mathrm{Re}(\{y_1(z) + i\epsilon[1 + O(\epsilon^2)]y_2(z)\} \times \exp i\{\omega + \epsilon n\pi[1 + O(\epsilon^2)]\}). \quad (57)$$

Thus the solution, which is $y_1(z)$ in $0 \leqslant z \leqslant \pi$, is obtained by taking $\omega = 0$. However, if this solution is taken to large values of z by (57) such that $n\epsilon \sim \frac{1}{2}$, then

$$y_H(z+n\pi) = -\epsilon y_2(z). \quad (58)$$

In this range $y(z+n\pi)$ vanishes near $z = 0$ and its magnitude is $O(\epsilon)$. Extensions of this argument show that every homogeneous solution of (37) has zeros and also that the order of magnitude of the solution can vary from $O(1)$ to $O(\epsilon)$ and back to $O(1)$. The order of magnitude refers to the magnitude of $y(z)$ on some full interval $z_0 \leqslant z \leqslant z_0 + \pi$ and not just at a point at which $y(z)$ may vanish.

We may now consider the nature of the general solution of (36). We add to $b_1(z)$ given by (44) a constant real multiple of $y_H(z)$ given by (56). This solution is clearly no longer periodic in z, but it is almost periodic with betatron oscillations of wavelength $2/\epsilon$. Provided $y_H(z) < (E_0^2 - 1)^{-3/2}\epsilon^{-2}$ everywhere, the composite solution has no zero crossings, so that the betatron oscillations do not cause breakdown of the cold fluid model. If $y_H(z) > (E_0^2 - 1)^{-3/2}\epsilon^{-2}$, then the solution is essentially the homogeneous solutions and the betatron oscillations cause orbit crossings and breakdown of the cold fluid model. We may reformulate the condition for a solution with betatron oscillations but without orbit crossings or breakdown of the cold fluid approximation as

$$n(z) > \tfrac{1}{2}B_w^2.$$

When

$$n(z) < \tfrac{1}{2}B_w^2,$$

the solution is essentially a homogeneous solution with betatron oscillations, orbit crossings, and breakdown of the cold fluid approximation. Even though the cold plasma solution may fail after many wiggler periods, it remains valid for as many wiggler periods as do occur before orbit crossing. In dimensional variables the condition is

$$\omega_p^2/\gamma \gtrless \tfrac{1}{2}[eB_w/(mc)]^2 = \tfrac{1}{2}(\Omega_c)^2,$$

where ω_p and Ω_c are the plasma frequency and cyclotron frequency in the laboratory frame of the electron fluid.

We observe that the wavelength of the betatron oscillations, $2\pi/\epsilon$, is independent of the density. We can present a simple physical explanation of this phenomenon based on the different roles of the wiggler field and of the self-fields of the beam. The wiggler field exerts a restoring force on the particles that is proportional to y and thus acts like the restoring force of a spring. The self-fields, on the other hand, exert a transverse force that is constant along streamlines. The nature of the self-field forces is easily understood when one notices that without crossings of streamlines the current and charge between a streamline and the axis of symmetry are constant in z, and thus the fields generated by the current and charge do not vary with z. Hence the particles are acted upon by a "spring force" (the wiggler) and a constant force (the self-fields). The constant force changes the equilibrium plane of the oscillation, but does not change the frequency of the oscillations. The equilibrium plane of the oscillation moves from the plane of symmetry to two symmetric planes on the two sides. This is in contrast to the betatron oscillations of a cylindrical beam in a helical wiggler, where the force exerted by the self-fields is not constant.[11] In the following we will demonstrate a case where the self-fields do modify the frequency of oscillations.

We next examine the Hamiltonian dynamics (22)–(24) in the paraxial approximation. We assume, for convenience, symmetry in y and we set

$$B_x = A_{z,y} = y\beta(z) + \cdots,$$
$$\phi = \phi_0(z) + (y^2/2)\phi_2(z) + \cdots,$$
$$\psi = \psi_0(z) + (y^2/2)\psi_2(z) + \cdots,$$

and we find

$$\Delta(z)\frac{d}{dz}\left(\Delta(z)\frac{dy}{dz}\right) + \{\Delta(z)\beta(z) - [H + \phi_0(z)]\phi_2(z) + \psi_2(z)[\psi_0(z) - p_x]\}y(z) = 0, \quad (59)$$

where here

$$\Delta^2(z) = [H - \phi_0(z)]^2 - 1 - [\psi_0(z) - p_x]^2, \quad (60)$$

and we have assumed p_y is small of order y in order to validate the paraxial approximation. The system (59) and (60) is not idential with (32) or (36), but it is generally quite similar, and the Floquet theory applies. In the extremely low density case we might drop the self-fields so that $\beta(z) = \phi_0(z) = \phi_2(z) = 0$, and we may select $p_x = 0$. If we take $\psi(y,z)$ to be the vacuum flux function $\psi = B_w \cosh y \cos z$, then

$$\psi_0(z) = \psi_2(z) = B_w \cos z, \quad (61)$$

and then (59) is identical to (37) and we may describe the single particle in terms of the earlier analysis. We note in passing that in this very low density case we can easily characterize two constants of the motion of the Hamiltonian system. These constants are

$$\Delta(z)[y'(z)y_1(z) - y(z)y_1'(z)] = C_1 \quad (62a)$$

and

$$\Delta(z)[y'(z)y_2(z) - y(z)y_2'(z)] = C_2, \quad (62b)$$

where $y_1(z)$ and $y_2(z)$ are the two fundamental solutions of (37) given by (41) and (42). These constants of the motion might be used as the basis for characterizing solutions of the Vlasov equation. They are not the usual type of integrals of the motion, since they involve the special functions $y_1(z)$ and $y_2(z)$. They are, nonetheless, constant in z.

If one does not assume that the self-field effects are negligible, then it is easy to select specific forms for $\beta(z)$, $\phi_0(z)$, and $\phi_2(z)$, such that the solutions of (59) are unstable. Instead of executing betatron oscillations, the particles would diverge to infinity or converge to zero as $|z|$ tends to infinity. A simple example would be

$$\phi_0(z) = \phi_2(z) = p_x = 0,$$
$$\psi_0(z) = \psi_2(z) = B_w \cos z,$$
$$\beta(z) = -B_w^2(1 + \cos^2 z),$$

for which (59) becomes

$$\Delta(z)\frac{d}{dz}\Delta(z)\frac{dy}{dz} - B_w^2 y = 0. \quad (63)$$

It is easy to show that for B_w/H small the solutions of (63) are all unstable. Thus it is clear that at low, but moderate, densities it is essential to include self-fields self-consistently in order to characterize the orbits meaningfully.

As a second example let us choose $\beta(z) = N(H^2 - 1)^{1/2}$ and $\phi_2(z) = NH$, where again $\phi_0 = 0$ and $\psi_0(z) = \psi_2(z) = B_w \cos z$, corresponding to the self-fields of a beam of density N which does not vary with z. If $\Delta^2(z) \cong H^2 - 1$, Eq. (59) becomes

$$\frac{d^2y}{dz^2} + \left(\frac{B_w^2}{2}(1 + \cos 2z) - N\right)y = 0. \quad (64)$$

This equation has oscillating solutions of the long space scale with frequency ω_b,

$$\omega_b^2 = B_w^2/2 - N.$$

The frequency ω_b is the well-known betatron oscillation frequency of a particle in the presence of a uniform magnetic field and a perpendicular electric field linear in the space coordinate. If N is larger than $B_w^2/2$ the motion is unbounded. We notice that a particle that moves inside an electron beam of constant density feels a varying electric field along its trajectory and the frequency of its oscillation is modified. Using the same terminology as in the discussion following Eq. (58), we may say that the self-fields in this configuration do not act as a constant force but rather as a spring opposing the direction of the wiggler field "spring." Thus the total spring constant is smaller and the betatron frequency is smaller. The different effects of the self-fields on the character of the betatron oscillation are likely to affect the FEL interaction.

IV. A THICK BEAM APPROXIMATION

We return to the cold fluid model presented in Sec. II and we look for an approximation scheme that admits of solutions with thick beams and that is not restricted to the paraxial approximation. We start from the system (17)–(20) and we expand in a formal small parameter ϵ. Many different scalings are possible, and we select one that appears to generate the most interesting family of solutions. The scaling simulates some of the effects of the paraxial approximation, but it clearly includes some finite thickness beam effects. We introduce a formal small parameter ϵ and we expand in the form

$$\chi = \epsilon\chi_1 + \epsilon^3\chi_3 + \epsilon^5\chi_5 + \cdots,$$
$$\psi = \epsilon\psi_1 + \epsilon^3\psi_3 + \epsilon^5\psi_5 + \cdots,$$
$$\phi = \epsilon\phi_1 + \epsilon^3\phi_3 + \epsilon^5\phi_5 + \cdots,$$

$$E = E_{-1}/\epsilon + E_1(\chi_1)\epsilon + [E_3(\chi_1) + E_1'(\chi_1)\chi_3]\epsilon^3 + \cdots,$$
$$F = F_1(\chi_1)\epsilon + [F_3(\chi_1) + F_1'(\chi_1)\chi_3]\epsilon^2 + \cdots,$$
$$h = \epsilon^2 h_2 + \epsilon^4 h_4 + \cdots.$$

The parameter ϵ measures the magnitude of $1/\gamma$, the square root of the density, and the square root of the ratio of transverse to longitudinal momenta. We wish to admit the possibility of axial variation on two distinct distance scales corresponding to the fast variation of the wiggler magnetic field and to the slow variation on the betatron oscillation wavelength. We assume that z corresponds to the fast wiggler scale, and we also introduce a "second" z coordinate

$$\zeta = \epsilon^2 z, \tag{65}$$

and we assume $\chi_1 = \chi_1(y,\zeta)$, $\chi_3 = \chi_3(y,\zeta)$, $\chi_5 = \chi_5(y,z,\zeta)$, etc.; $\phi = \phi(y,z,\zeta)$ and $\psi = \psi(y,z,\zeta)$. We now expand (17)–(20) order by order. In the process it is convenient to expand in the order, first (17), then (19) and (20), and finally (18). We are particularly interested in solutions that are periodic of period L on the fast (z) distance scale and we wish also to examine the behavior of solutions on the slow ($\zeta = \epsilon^2 z$) distance scale. It is convenient to introduce the notation

$$\langle f,(y,z,\zeta)\rangle = \frac{1}{L}\int_0^L f(y,z,\zeta)dz, \tag{66}$$

so that $\langle f\rangle$ is the average of f on the fast (z) distance scale. Further, we define

$$\tilde{f}(y,z,\zeta) = f - \langle f\rangle, \tag{67}$$

and

$$\langle \tilde{f}\rangle = 0.$$

In the lowest order we find

$$h_2 = \chi_{1,y}/E_{-1}, \tag{68}$$
$$\Delta\phi_1 = \chi_{1,y}, \tag{69}$$
$$\Delta\psi_1 = 0, \tag{70}$$

and

$$\chi_{1,y}^2\chi_{3,zz} = -h_2(\chi_{1,y})^2(\phi_{1,y} - \chi_1). \tag{71}$$

From (69) we see that

$$\phi_1 = \langle\phi_1\rangle + \tilde{\phi}_1, \tag{72}$$

where

$$\langle\phi_{1,y}\rangle = \chi_1 \tag{73}$$

and

$$\Delta\tilde{\phi}_1 = 0. \tag{74}$$

Thus $\tilde{\phi}_1$ is a vacuum electrostatic potential, $\psi_1 = \tilde{\psi}_1$ is a vacuum magnetic flux function, and

$$\chi_{3,zz} = -\chi_{1,y}\tilde{\phi}_{1,y}/E_{-1}. \tag{75}$$

Since $\tilde{\phi}_1$ has zero average in z we may integrate (75) to obtain χ_3 as a function periodic of period L in z, or

$$\chi_3 = \langle\chi_3\rangle + \tilde{\chi}_3, \tag{76}$$

where

$$\tilde{\chi}_{3,zz} = -\chi_{1,y}\tilde{\phi}_{1,y}/E_{-1}, \tag{77}$$

and $\langle\chi_3\rangle$ is a function of y and ζ only.

In next order we find

$$h_2 h_4 E_{-1}^2 + h_2^2 E_{-1}(E_1 + \phi_1) - \chi_{1,y}\chi_{3,y} = \tfrac{1}{2}h_2^2,$$

or, with (68),

$$h_4 E_{-1} + (E_1 + \phi_1)h_2 = \chi_{3,y} + \tfrac{1}{2}\chi_{1,y}/E_{-1}^2. \tag{78}$$

Poisson's equation, (19), becomes

$$\Delta\phi_3 = \chi_{3,y} + \tfrac{1}{2}\chi_{1,y}/E_{-1}^2$$

and

$$\phi_3 = \langle\phi_3\rangle + \tilde{\phi}_3, \tag{79}$$

where

$$\langle\phi_3\rangle_{,y} = \langle\chi_3\rangle + \tfrac{1}{2}\chi_1/E_{-1}^2, \tag{80}$$
$$\Delta\tilde{\phi}_3 = \tilde{\chi}_{3,y}, \tag{81}$$

while (20) becomes

$$\Delta\psi_3 = h_2[\psi_1 - F_1(\chi_1)]. \tag{82}$$

The remaining equation, (18), is

$$\begin{aligned}\chi_{1,y}^2\chi_{5,zz} &+ \chi_{1,y}^2(\chi_{3,\zeta z} + \chi_{1,\zeta\zeta}) + 2\chi_{1,y}\chi_{3,y}(\chi_{1,\zeta z} + \chi_{3,zz}) \\ &- 2\chi_{1,y}(\chi_{3,z} + \chi_{1,\zeta})(\chi_{3,z} + \chi_{1,\zeta})_{,y} \\ &+ (\chi_{3,z} + \chi_{1,\zeta})^2\zeta_{1,yy} \\ &= -h_4\chi_{1,y}^2\tilde{\phi}_{1,y} - h_2\{[h_4E_{-1} + h_2(E_1 + \phi_1)]\chi_{1,y}\phi_{1,y} \\ &\quad + h_2E_{-1}(\phi_{3,y}\chi_{1,y} + \phi_{1,y}\chi_{3,y}) \\ &\quad - \chi_3\chi_{1,y}^2 - 2\chi_1\chi_{1,y}\chi_{3,y} - h_2[\psi_1 - F(\chi_1)]\psi_{1,y}\chi_{1,y}\},\end{aligned}$$

or

$$\begin{aligned}\chi_{1,y}^2\chi_{5,zz} &+ \chi_{1,y}^2\mu_{,\zeta} + 2\chi_{1,y}\chi_{3,y}\mu_{,z} - 2\chi_{1,y}\mu\mu_{,y} + \mu^2\chi_{1,yy} \\ &= -\frac{\chi_{1,y}^2}{E_{-1}}\bigg[\tilde{\phi}_{1,y}\bigg(3\chi_{3,y} + \frac{1}{2}\frac{\chi_{1,y}}{E_{-1}^2} - \frac{(E_1 + \phi_1)\chi_{1,y}}{E_{-1}}\bigg) \\ &\quad + \chi_{1,\zeta}\phi_{1,z} + \chi_{1,y}\bigg(\tilde{\phi}_{3,y} - \tilde{\chi}_3 + \frac{\tilde{\phi}_{1,y}}{E_{-1}^2}\bigg) \\ &\quad + \frac{\chi_1\chi_{1,y}}{E_{-1}^2} - \frac{\chi_{1,y}\psi_{1,y}[\psi_1 - F_1(\chi_1)]}{E_{-1}}\bigg],\end{aligned} \tag{83}$$

where

$$\mu = \chi_{3,z} + \chi_{1,\zeta}. \tag{84}$$

We ask first whether there are any solutions of (83) periodic in z, that is, solutions with no ζ dependence. If we take the average of (83) with respect to z we obtain the condition for the existence of such solutions, namely,

$$\begin{aligned}2\chi_{1,y}\langle\tilde{\chi}_{3,y}\tilde{\chi}_{3,zz}\rangle &- 2\chi_{1,y}\langle\tilde{\chi}_{3,z}\tilde{\chi}_{3,zy}\rangle + \langle(\tilde{\chi}_{3,z})^2\rangle\chi_{1,yy} \\ &= -\frac{\chi_{1,y}^2}{E_{-1}}\bigg[3\langle\tilde{\phi}_{1,y}\tilde{\chi}_{3,y}\rangle - \frac{\langle\tilde{\phi}_{1,y}\tilde{\phi}_1\rangle\chi_{1,y}}{E_{-1}} \\ &\quad + \frac{\chi_{1,y}}{E_{-1}}\bigg(\frac{\chi_1}{E_{-1}} - \langle\tilde{\psi}_1\tilde{\psi}_{1,y}\rangle\bigg)\bigg].\end{aligned} \tag{85}$$

If (85) is satisfied then there is a solution of (83) with no ζ dependence of χ_1, χ_3, or χ_5, and which is periodic in z of period L. With the identities

$$\langle\tilde{\chi}_{3,y}\tilde{\chi}_{3,zz}\rangle = -\chi_{1,y}\langle\tilde{\phi}_{1,y}\tilde{\chi}_{3,y}\rangle,$$
$$\langle\tilde{\chi}_{3,z}\tilde{\chi}_{3,zy}\rangle = \chi_{1,y}\langle\tilde{\phi}_{1,y}\tilde{\chi}_{3,y}\rangle/E_{-1},$$
$$\langle\tilde{\chi}_{3,z}\tilde{\chi}_{3,z}\rangle = \langle\tilde{\chi}_3\tilde{\phi}_{1,y}\rangle\chi_1/E_{-1},$$
$$\langle\tilde{\phi}_{1,yy}\tilde{\chi}_3\rangle = -\langle\tilde{\phi}_{1,zz}\tilde{\chi}_3\rangle = \langle\tilde{\phi}_1\tilde{\phi}_{1,y}\rangle\chi_1/E_{-1},$$

obtained from (74), (77), and integration by parts, we find easily that (85) reduces to

$$(\langle\tilde{\phi}_{1,y}\tilde{\chi}_3\rangle/\chi_{1,y,y} = (\chi_1/E_{-1} - \langle\tilde{\psi}_1\tilde{\psi}_{1,y}\rangle)/E_{-1}. \qquad (86)$$

For the external electric and magnetic fields corresponding to

$$\tilde{\psi}_1 = (\alpha e^{y} + \beta e^{-y})\cos z, \qquad (87a)$$

$$\tilde{\phi}_1 = (\lambda e^{y} + \mu e^{-y})\cos z, \qquad (87b)$$

we obtain easily

$$\tilde{\chi}_3 = (\chi_{1,y}/E_{-1})(\lambda e^{y} - \mu e^{-y})\cos z$$

and

$$\langle\tilde{\phi}_{1,y}\tilde{\chi}_3\rangle/\chi_{1,y} = \tfrac{1}{2}(\lambda e^{y} - \mu e^{-y})^2/E_1,$$

so that

$$\chi_1/E_{-1} = (\lambda^2 e^{2y} - \mu^2 e^{-2y}) + \tfrac{1}{2}(\alpha^2 e^{2y} - \beta^2 e^{-2y}), \qquad (88)$$

and

$$h_2(y) = 2(\lambda^2 e^{2y} + \mu^2 e^{-2y}) + (\alpha^2 e^{2y} + \beta^2 e^{-2y}). \qquad (89)$$

Thus the only density profiles that generate flows of the same periodicity as the applied wiggler fields are given by (89). These profiles are hollow as the largest densities occur at the edges of the beam. The function $h_2(y)$ has at most one minimum and $h_2(y)$ increases monotonically and exponentially away from that minimum. The usual case with symmetry in y occurs for $\alpha = \beta = B_w/2$ and $\lambda = \mu = E_w/2$, for which

$$\tilde{\psi}_1 = B_w \cosh y \cos z,$$

$$\tilde{\phi}_1 = E_w \cosh y \cos z,$$

and

$$h_2(y) = (B_w^2 + \tfrac{1}{2}E_w^2)\cosh 2y.$$

It is clear that we could carry out this perturbation expansion order by order in ϵ and construct a formal solution periodic in z. We cannot address, however, the much deeper question of whether or not exact periodic solutions of our system exist. In any case, we have shown that within the perturbation expansion we can construct solutions not affected by betatron oscillations.

We also consider one final question in order to gain some insight as to the nature of the solutions of (83). We examine the family of solutions in the neighborhood of the exactly periodic solutions just found. For simplicity we restrict ourselves to the case with no externally applied vacuum electric field, $\tilde{\phi}_1 = 0$ and $F_1(\chi_1) = 0$, so that $\tilde{\chi}_3 = \tilde{\phi}_3 = 0$. Further, we take the simple solution corresponding to

$$\tilde{\psi}_1 = B_w \cosh y \cos z, \qquad (90)$$

for which the periodic solution of (83) is

$$\bar{\chi}_1/E_{-1} = \tfrac{1}{2}B_w^2 \sinh 2y \qquad (91)$$

and

$$E_{-1}^2\bar{\chi}_5 = -\tfrac{1}{8}B_w^2\bar{\chi}_{1,y} \cosh y \sinh y \cos 2z, \qquad (92)$$

while

$$\bar{h}_2(y) = B_w^2 \cosh 2y. \qquad (93)$$

We seek solutions near $\bar{\chi}_1$ and $\bar{\chi}_5$ and set

$$\chi_1 = \bar{\chi}_1 + \delta\chi_1,$$

$$\chi_5 = \bar{\chi}_5 + \delta\chi_5,$$

where we assume $\delta\chi_1$ and $\delta\chi_5$ are small. We find that to lowest order in $\delta\chi_1$ and $\delta\chi_5$,

$$\bar{\chi}_{1,y}^2\delta\chi_{1,\zeta\zeta} + 2\bar{\chi}_{1,y}\delta\chi_{1,y}\bar{\chi}_{5,zz} + \bar{\chi}_{1,y}^2\delta\chi_{5,zz} = -(\bar{\chi}_{1,y}/E_{-1})^3\delta\chi_1. \qquad (94)$$

We may solve (94) for $\delta\chi_5$ as a function periodic of the fast period provided the average of (94) over the fast period vanishes. Thus we require

$$\delta\chi_{1,\zeta\zeta} + (\bar{\chi}_{1,y}/E_{-1}^3)\delta\chi_1 = 0, \qquad (95)$$

or

$$\delta\chi_1 = A(y)\cos\{\zeta[h_2(y)]^{1/2}/E_{-1}\} + B(y)\sin\{\zeta[h_2(y)]^{1/2}/E_{-1}\}. \qquad (96)$$

The solutions near the exact periodic solution are bounded and exhibit betatron oscillations of wavenumber in z equal to $\epsilon^2[h_2(y)]^{1/2}/E_{-1}$. Hence the oscillation period is different on each streamline. Thus even though we may find betatron oscillations, they do not necessarily signal the failure of the cold fluid model.

We cannot guarantee that the solutions exhibiting betatron oscillations are uniformly bounded or that no orbit crossings occur. Neither can we be sure that exact periodic solutions exist. A direct extension of our analysis would show that there is a formal expansion that satisfies the equations to all orders and that is periodic to all orders. We must leave open the more difficult questions concerning the properties of exact solutions.

V. SUMMARY

In this paper a general planar relativistic non-neutral steady cold fluid model was formulated. It was applied to the study of the equilibrium of a sheet electron beam in a planar wiggler FEL. The full transverse dependence of the wiggler field as well as the equilibrium self-fields of the beam were taken into account. In several cases the self-fields were shown to have a significant effect on the equilibrium. For a thick beam we found a particular density profile such that the wiggler focusing is balanced by the self-fields, and the betatron oscillations disappear. In the paraxial approximation we also found a critical density for which there are no betatron oscillations. If the density is larger than this critical density the beam oscillates with the betatron frequency but its thickness does not tend to zero. Here, as a result of the self-fields, no orbit crossing occurs. Such an equilibrium could be of interest for FEL applications. The single-particle equations of motion were examined in the presence of self-fields and constants of motion were identified. If the equilibrium is such that the self-fields do not vary along z, the particles were shown to execute betatron oscillations with a reduced frequency because of the self-fields. The effect of these particle motions on the FEL could be of interest. We note, in addition, that in the paraxial approximation for both the cold fluid and the single-particle motion, the parallel momentum is not constant when self-fields are present.

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