A thick beam free electron laser

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A 3-D theory is presented for a free electron laser that employs an electron beam of a thickness comparable to both the wiggler wavelength and the waveguide radius. The time-independent and the linearized time-dependent cold fluid and Maxwell equations are expanded in a small parameter, which is the ratio of the perpendicular to parallel electron momentum. The stability problem is reduced to a nonlinear eigenvalue problem of a fourth-order system of linear ordinary differential equations. A perturbation method is justified and used to solve these equations. A dispersion relation is derived which results from the solvability condition for the first-order equations in the perturbation. The orders of magnitude of the beam density and wave frequency, for which the growth rate of the instability scales as in the strong-pump regime of the 1-D analysis, are determined. An equation, which the beam energy radial profile has to satisfy, is also derived.

I. INTRODUCTION

In the free electron laser (FEL), which is a source of tunable high-intensity electromagnetic radiation,[1] a relativistic electron beam amplifies electromagnetic waves. In order to maximize the power generated by the free electron laser, it is desirable to have a large number of resonant electrons in the form of a high-current beam. In practice, the way to increase the current is to have a large cross-section beam. It is our purpose here to study a free electron laser that employs a thick beam. Contrary to previous 3-D theories that dealt with thin beams[2] or filament-type beams,[3] we consider a beam with a radius comparable to the wavelength of the wiggler and the radius of the waveguide. We find a parameter regime in which the whole beam is resonant and the growth rate scales the same as in the approximated 1-D theory of the strong-pump regime,[4] in which it is proportional to the cubic root of the product of the density and the square of the perpendicular velocity amplitude. In order to preserve this scaling the parallel velocity of the beam has to be uniform across the beam to a high degree. The radial gradients of the wiggler field and the equilibrium self-fields create shear in this parallel velocity. We determine the orders of magnitude of the beam density and wave frequency that are needed in order to preserve the scaling of the growth rate. We also derive an equation that the beam energy radial profile has to satisfy. In this case, the degradation and limitations of FEL performance that the shear implies, which have been well recognized in the 1-D theory,[5] do not affect FEL operation to lowest order.

The equilibrium, of which we study the stability, is that of a relativistic helically symmetric cold electron fluid in the presence of both uniform guide magnetic field and helical wiggler field.[6] The wiggler field, and correspondingly the flow, have a general helical magnetic multipole number and are not limited to the usual dipole moment or to the quadrupole moment studied recently.[7] Self-fields are also taken into account. The time-independent equations are expanded in the small ratio of perpendicular to parallel velocities. The requirement of low shear dictates the use of a tenuous beam so that to lowest order the self-fields do not appear.

We perform a stability analysis by solving the linearized time-dependent cold fluid equations and the Maxwell equations. We now expand the linearized equations in the small ratio of perpendicular to parallel velocities. The beam couples the various helical harmonics of the wave. However, when one of the harmonics becomes large, the series of infinite coupled equations can be truncated. To lowest order in the expansion we derive a fourth-order system of ordinary differential equations of a nonlinear eigenvalue problem. In the complex eigenvalue plane there are infinite nonreal eigenvalues in the neighborhood of zero. By adjusting the equilibrium quantities or the frequency, one of the modes may satisfy a resonance condition typical of free electron lasers. Near that resonance the eigenvalue of the unstable resonant mode becomes large and the mode becomes, to lowest order, a vacuum waveguide mode. In this case, the equations are solved using a perturbation method. The eigenvalue is found by calculating the roots of the dispersion relation that results from the solvability condition. The dispersion relation is a cubic polynomial of a form familiar from 1-D analyses of free electron lasers in the strong-pump regime.

In Sec. II the thick beam equilibrium is described. In Sec. III the linearized time-dependent cold fluid and Maxwell equations are presented, the fluid equations are approximated for the beam modes, and a simple expression for the perturbed density is derived. The fluid equations are coupled to Maxwell's equations to yield the final fourth-order system of equations. In Sec. IV the domain in the complex plane, where nonreal eigenvalues are possible, is found and a dispersion relation is derived for the resonant case. The nonresonant unstable modes are also discussed in Sec. IV and we conclude in Sec. V.

II. THE EQUILIBRIUM

We consider a helical flow, which depends on r and ϕ ($=\theta - kz$), where r, θ, and z are the usual cylindrical co-

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ordinates, and which is characterized by normalized density and momentum. The normalized density h is

$$h=\omega_p^2/c^2\gamma, \qquad (1)$$

where ω_p is the plasma frequency ($\omega_p^2=4\pi Ne^2/m$), N is the electron density, e and m are the electron charge and mass, $(\gamma-1)mc^2$ is the electron kinetic energy, and c is the velocity of light in vacuum. The cylindrical components of $\mathbf{u}$ (the momentum divided by mc) are u, v, and w. The helical flow is driven by an M multiple external magnetic helical wiggler field and is confined by a uniform axial magnetic field.

As in Ref. 6, we expand the equilibrium quantities in the small parameter ϵ, which measures the ratio of the magnitudes of the perpendicular and parallel momentum components, as well as the ratio of the magnitudes of the external wiggler and uniform axial magnetic fields. Since we want the parallel velocity of the beam to be uniform to some high order, we limit the density to second order in ϵ. We assume also that to lowest order we have a straight beam with uniform parallel momentum and no perpendicular momentum. The energy of the beam, which is constant to lowest order, is

$$\gamma_0=O(\epsilon^{-p_\gamma}),\quad p_\gamma\geqslant 0. \qquad (2)$$

This choice of low density, vacuum fields, and beam profile was shown in Ref. 6 to result in the following forms:

of the density,

$$h=\epsilon^2 h_2(r)+O(\epsilon^3); \qquad (3)$$

of the momenta,

$$\epsilon^{p_\gamma}u=\epsilon(MA/kr)\{\alpha r[I_M(Mkr)]_{,r}+M^2 I_M(Mkr)\}\times[(M^2-\alpha^2)\cos M\phi]^{-1}+O(\epsilon^2), \qquad (4a)$$

$$\epsilon^{p_\gamma}v=-\epsilon(M^2A/kr)\{r[I_M(Mkr)]_{,r}+\alpha I_M(Mkr)\}\times[(M^2-\alpha^2)\sin M\phi]^{-1}+O(\epsilon^2), \qquad (4b)$$

$$w=w_0+w_1(r)+O(\epsilon^{2-p_\gamma}); \qquad (4c)$$

of the energy,

$$\gamma=\gamma_0+\gamma_1(r)+O(\epsilon^{2-p_\gamma}); \qquad (5)$$

and of the fields,

$$\mathbf{E}=O(\epsilon^{2-p_\gamma}), \qquad (6a)$$

$$\epsilon^{p_\gamma}B_r=-\epsilon AM[I_M(Mkr)]_{,r}\cos M\phi+O(\epsilon^2), \qquad (6b)$$

$$\epsilon^{p_\gamma}B_\theta=[\epsilon A/(1+k^2r^2)]\{r[I_M(Mkr)]_{,r}\}_{,r}\times\sin M\phi+O(\epsilon^2), \qquad (6c)$$

$$\epsilon^{p_\gamma}B_z=\epsilon^{p_\gamma}B_0-[\epsilon Akr/(1+k^2r^2)]\{r[I_M(Mkr)]_{,r}\}_{,r}\times\sin M\phi+O(\epsilon^2). \qquad (6d)$$

Here

$$\mathbf{E}=e\mathbf{E}'/mc^2, \qquad (7a)$$

$$\mathbf{B}=e\mathbf{B}'/mc^2, \qquad (7b)$$

and $\mathbf{E}'$ and $\mathbf{B}'$ are the electric and magnetic fields, B_0 is the magnitude of the normalized guide field, and

$$\alpha\equiv B_0/(kw_0). \qquad (8)$$

Note that γ_0, w_0, and B_0 are $O(\epsilon^{-p_\gamma})$ and A is $O(1)$. The equilibrium self-fields of the beam and the flow induced by these fields are of second order only. In Sec. III we perform a stability analysis of the equilibrium. We show that the knowledge of the equilibrium quantities is needed to first order only. Thus in our case the equilibrium self-fields are neglected. In a following paper we will study a case where these self-fields play a major role.

III. THE LINEARIZED EQUATIONS

The perturbed density $\delta h'$, momenta $\delta\mathbf{u}'$, and energy $\delta\gamma'$ of the beam are governed by the linearized continuity equation

$$\frac{\partial}{\partial t}(\gamma\delta h'+h\delta\gamma')+\nabla\cdot(h\,\delta\mathbf{u}'+\delta h'\,\mathbf{u})=0, \qquad (9)$$

the linearized momenta equation

$$\gamma\frac{\partial}{\partial t}\delta\mathbf{u}'+\mathbf{u}\cdot\nabla\,\delta\mathbf{u}'+\delta\mathbf{u}'\cdot\nabla\mathbf{u}=-\gamma\,\delta\mathbf{E}'-\delta\gamma'\,\mathbf{E}-\delta\mathbf{u}'\times\mathbf{B}-\mathbf{u}\times\delta\mathbf{B}', \qquad (10)$$

and the relation

$$\gamma\,\delta\gamma'=\mathbf{u}\cdot\delta\mathbf{u}'. \qquad (11)$$

The perturbed wave electric field $\delta\mathbf{E}'$ and magnetic field $\delta\mathbf{B}'$ are governed by the linearized Maxwell equations

$$\nabla\times\nabla\times\delta\mathbf{E}'+\frac{\partial^2}{\partial t^2}\delta\mathbf{E}'=\frac{\partial}{\partial t}(\delta h'\,\mathbf{u}+h\,\delta\mathbf{u}') \qquad (12)$$

and

$$\frac{\partial\delta\mathbf{B}'}{\partial t}=-\nabla\times\delta\mathbf{E}'. \qquad (13)$$

Since the equilibrium does not depend explicitly on z and since there has to be periodic dependence of ϕ, we seek a solution for the perturbed quantities of the form

$$f'=\left(\sum_{l=-\infty}^{\infty}f^{(l)}(r)\exp(il\phi)\right)\exp[i(qz-\omega t)]. \qquad (14)$$

We limit ourselves to the case of FEL resonance, so that

$$p_\omega=2p_\gamma, \qquad (15)$$

where

$$\omega=O(\epsilon^{-p_\omega}). \qquad (16)$$

The eigenvalue q is assumed to be of the same order of magnitude as ω:

$$q=O(\epsilon^{-p_\omega}). \qquad (17)$$

We write every equilibrium quantity A to lowest order in the form

$$A=A^{(M)}e^{iM\phi}+A^{(-M)}e^{-iM\phi}+A^{(0)}, \qquad (18)$$

where $A^{(M)}$, $A^{(-M)}$, and $A^{(0)}$ are functions of r. Inserting form (14) for the perturbed quantities and form (18) for the equilibrium quantities into the fluid and Maxwell equations, we obtain an infinite set of coupled ordinary differential equations. The l*th harmonic of the continuity equation is

$$\sum_{N=0,\pm M}\left[i\left(-\omega\gamma^{(N)}+\frac{l^*}{r}v^{(N)}+(q-l^*k)w^{(N)}\right)+\frac{1}{r}\frac{\partial}{\partial r}(ru^{(N)})+u^{(N)}\frac{\partial}{\partial r}\right]\delta h^{(l^*-N)}$$
$$=-\sum_{N=0,\pm M}ih^{(N)}\left(-\omega\,\delta\gamma^{(l^*-N)}+\frac{l^*}{r}\delta u^{(l^*-N)}+(q-l^*k)\delta w^{(l^*-N)}\right)-\sum_{N=0,\pm M}\frac{1}{r}\frac{\partial}{\partial r}[rh^{(N)}\,\delta u^{(l^*-N)}]. \tag{19}$$

The l^*th harmonic of the z component of the momentum equation is

$$\sum_{N=0,\pm M}\left[i\left(-\omega\gamma^{(N)}+\frac{(l^*-N)}{r}v^{(N)}+(q-kl^*+kN)w^{(N)}\right)+\frac{\partial u^{(N)}}{\partial r}+u^{(N)}\frac{\partial}{\partial r}\right]\delta w^{(l^*-N)}$$
$$=\sum_{N=0,\pm M}[-\gamma^{(N)}\,\delta E_z^{(l^*-N)}-E_z^{(N)}\,\delta\gamma^{(l^*-N)}-B_\theta^{(N)}\,\delta u^{(l^*-N)}$$
$$+B_r^{(N)}\,\delta v^{(l^*-N)}-u^{(N)}\,\delta B_\theta^{(l^*-N)}+v^{(N)}\,\delta B_r^{(l^*-N)}]. \tag{20}$$

The perpendicular components of the momentum equation are

$$\sum_{N=0,\pm M}\left\{i\left[-\omega\gamma^{(N)}+\left(\frac{l^*-N\mp 1}{r}\right)v^{(N)}+[q-k(l^*-N)]w^{(N)}\mp B_z^{(N)}\right]+\frac{\partial u^{(N)}}{\partial r}+u^{(N)}\frac{\partial}{\partial r}\right\}$$
$$\times(\delta u^{(l^*-N)}\pm i\delta v^{(l^*-N)})$$
$$=\sum_{N=0,\pm M}[-\gamma^{(N)}(\delta E_r^{(l^*-N)}\pm i\,\delta E_\theta^{(l^*-N)})-(E_r^{(N)}\pm iE_\theta^{(N)})\delta\gamma^{(l^*-N)}$$
$$\mp i(B_r^{(N)}\pm iB_\theta^{(N)})\delta w^{(l^*-N)}\pm i(u^{(N)}\pm iv^{(N)})\delta B_z^{(l^*-N)}\mp iw^{(N)}(\delta B_r^{(l^*-N)}\pm i\delta B_\theta^{(l^*-N)})]. \tag{21}$$

Finally, the l^*th harmonics of the Maxwell equations are

$$\frac{\partial^2}{\partial r^2}(r\delta E_r^{(l^*)}\pm ir\delta E_\theta^{(l^*)})-\frac{1}{r}\frac{\partial}{\partial r}(r\delta E_r^{(l^*)}\pm ir\delta E_\theta^{(l^*)})+\left(\omega^2-(q-l^*k)^2-\frac{l^*}{r^2}(l^*\pm 2)\right)(r\delta E_r^{(l^*)}\pm ir\delta E_\theta^{(l^*)})$$
$$=S_\pm^{(l^*)}\equiv\sum_{N=0,\pm M}\Big[i\omega(u^{(N)}\pm iv^{(N)})\delta h^{(l^*-N)}+i\omega h^{(N)}(\delta u^{(l^*-N)}\pm i\delta v^{(l^*-N)})$$
$$+\left(-\frac{\partial}{\partial r}-\frac{il^*}{r}\right)(\gamma^{(N)}\delta h^{(l^*-N)}+h^{(N)}\delta\gamma^{(l^*-N)})\Big]. \tag{22}$$

Solving this infinite set of coupled differential equations is a formidable task. However, we are interested only in the case in which some of the quantities become large and the equations describing them decouple from the rest of the equations and comprise a truncated finite set of equations. These quantities are $\delta\mathbf{E}^{(l^*)}$, $\delta\mathbf{B}^{(l^*)}$, $\delta w^{(l^*-M)}$, and $\delta h^{(l^*-M)}$ for l^* equals some l. The magnetic field $\delta\mathbf{B}^{(l)}$ and the helical flow $\mathbf{u}^{(-M)}$ create a ponderomotive force that causes large longitudinal bunching and makes a large parallel momentum $\delta w^{(l-M)}$. The parallel momentum $\delta w^{(l-M)}$ and the perturbed density $\delta h^{(l-M)}$ are coupled through the continuity equation. Finally, the product of the large $\delta h^{(l-M)}$ and the helical flow contributes to the perpendicular current, which is then coupled to $\delta\mathbf{B}^{(l)}$ through Maxwell's equations. For a special choice of parameter values, the quantities just mentioned become dominant, as they are in the 1-D theory of the FEL in the strong-pump regime. Thus we intend to determine under which conditions the basic interaction of the thick beam FEL is the same as the interaction of an idealized 1-D FEL in the strong-pump regime.

The component $\delta w^{(l-M)}$ is large if its coefficient in the momentum equation (20) for $l^*=l-M$ is small. We define

$$\nu w_0\equiv(q-kl+kM)w_0-\omega\gamma_0=O(\epsilon^{p_q-p_\gamma}) \tag{23}$$

and look for the case in which

$$p_q>0, \tag{24}$$

which insures a small ν. This case corresponds to the interaction of the wave with a beam mode via the wiggler field. We note that for every frequency ω there is an eigenvalue q which satisfies condition (24). Later, when we derive a finite set of coupled equations, we obtain an additional relation between ω and q. These two relations will determine ω and q to lowest order, and ω will be found to be the well-known Doppler-shifted resonant frequency. By requiring that ν is small we are aiming our analysis at the strong-pump regime. Were we to explore the Raman regime, we would have to require that a different parameter ν_R be small[8]:

$$\nu_R=\nu+\omega_p\gamma_0/w_0. \tag{25}$$

Returning now to Eq. (20), we find that the coefficient of $\delta w^{(l-M)}$ is

$$\nu w^{(0)}+(l-M)v^{(0)}/r. \tag{26}$$

For the beam to be entirely resonant, or for the coefficient to be small for every r, we require

$$\nu w^{(0)}\gg(l-M)v^{(0)}/r, \tag{27}$$

which will be satisfied according to Eq. (4) if

$$p_q<2. \tag{28}$$

By using the smallness of ν and the relations between the zeroth-, first-, and second-order expressions for γ, u, v, and w, which follow the identity

$$\gamma^2 = 1 + u^2 + v^2 + w^2, \tag{29}$$

we find that the coefficient of $\delta w^{(l-M)}$, to lowest order, is

$$\begin{aligned} & -\omega\gamma^{(0)} + (q - kl + kM)w^{(0)} \\ & \quad = \nu w_0 + (\omega/w_0^2)\gamma_1 - (2\omega\gamma_0/w_0^2) \\ & \qquad \times (u^{(M)}u^{(-M)} + v^{(M)}v^{(-M)}). \end{aligned} \tag{30}$$

We would like the first term on the right-hand side (rhs) to be the largest. However, it can be shown that in the strong-pump regime, the third term, which is $O(\epsilon^{2-3p_\gamma})$, is larger than the first term, which is $O(\epsilon^{p_q - p_\gamma})$. The only way to insure the dominance of νw_0 on the rhs of Eq. (30) is to require that the second and third terms cancel each other. Therefore, the beam profile has to obey

$$\gamma_1 - 2\gamma_0(u^{(M)}u^{(-M)} + v^{(M)}v^{(-M)}) = 0. \tag{31}$$

The second term is $O(\epsilon^{2-3p_\gamma})$ and γ_1 is $O(\epsilon^{1-p_\gamma})$. Thus

$$p_\gamma = \tfrac{1}{2} \tag{32}$$

and

$$p_\omega = 1. \tag{33}$$

Equations (31)–(33) constitute some of the main results of this work. In order for a thick beam to be wholly resonant, with the resonance parameter of the strong-pump regime, it is necessary for its density h to be $O(\epsilon^2)$; its energy to be $O(\epsilon^{-1/2})$; the wave frequency ω to be $O(\epsilon^{-1})$; and the beam energy to be of a particular profile, described by Eq. (31).

Having specified the orders of γ and ω and using the inequality $p_q < 2$, we are able to simplify the equation for $\delta w^{(l-M)}$ much further. This equation decouples now from the equations for $\delta w^{(l)}$ and $\delta w^{(l-2M)}$ and becomes

$$\begin{aligned} & i\nu w_0\, \delta w^{(l-M)} \\ & \quad = \sum_{N=\pm M} (-B_\theta^{(N)}\, \delta u^{(l-M-N)} + B_r^{(N)}\, \delta v^{(l-M-N)}) \\ & \qquad - u^{(-M)}\, \delta B_\theta^{(l)} + v^{(-M)}\, \delta B_r^{(l)} - \gamma_0\, \delta E_z^{(l-M)}. \end{aligned} \tag{34}$$

In our case it can be shown that the first term on the rhs of Eq. (34) is relatively small and thus

$$\begin{aligned} i\nu w_0\, \delta w^{(l-M)} = & -(\gamma_0/w_0)(\delta E_r^{(l)} u^{(-M)} + \delta E_\theta^{(l)}\, v^{(-M)}) \\ & - \gamma^{(0)}\, \delta E_z^{(l-M)}. \end{aligned} \tag{35}$$

This final form of the equation was a result of the coupling of $\delta w^{(l-M)}$ and $\delta \mathbf{E}^{(l)}$ by the ponderomotive force. Because the coefficient ν is small, $\delta w^{(l-M)}$ is large, which corresponds to large longitudinal bunching.

We now turn to the continuity equation. On the rhs of the continuity equation (19), the largest terms are those containing $\delta w^{(l-M)}$ and $\delta\gamma^{(l-M)}$, which are $O(\epsilon^{1-p_q})$. Since $p_q < 2$, the equation for $\delta h^{(l-M)}$ becomes, to lowest order,

$$\begin{aligned} \nu w_0 \delta h^{(l-M)} = & -h^{(0)}\{-\omega\delta\gamma^{(l-M)} \\ & + [q - (l-M)k]\delta w^{(l-M)}\}, \end{aligned} \tag{36}$$

where $h^{(0)}$ is $\epsilon^2 h_2$ of Eq. (3). We use relation (11) to obtain

$$\gamma_0 \delta\gamma^{(l-M)} = w_0 \delta w^{(l-M)} \tag{37}$$

and thus

$$\nu w_0\, \delta h^{(l-M)} = -(h^{(0)}\omega/\gamma_0 w_0)\delta w^{(l-M)}. \tag{38}$$

The large parallel momentum $\delta w^{(l-M)}$ creates large density modulation $\delta h^{(l-M)}$. The density modulation is large also because its coefficient in Eq. (38) is ν.

Having an expression for $\delta h^{(l-M)}$ we are now ready to estimate the magnitude of $\delta E_z^{(l-M)}$. For Gauss' law we have

$$\begin{aligned} & \frac{1}{r}\frac{\partial}{\partial r}(r\,\delta E_r^{(l-M)}) + \frac{i(l-M)}{r}\,\delta E_\theta^{(l-M)} \\ & \quad + i(q - lk)\delta E_z^{(l-M)} \\ & \quad = -\gamma^{(0)}\,\delta h^{(l-M)}. \end{aligned} \tag{39}$$

We may neglect the term $-\gamma_0\,\delta E_z^{(l-M)}$ in Eq. (35) if $\delta E_z^{(l-M)}$ is $o(\epsilon)$. Because of Eqs. (35), (38), and (39), $\delta E_z^{(l-M)}$ is $O(\epsilon^{4-2p_q})$. We require

$$3 - 2p_q > 0. \tag{40}$$

This inequality will be shown shortly to be satisfied. The omission of $\delta E_z^{(l-M)}$ in Eq. (35) corresponds to the strong-pump regime as opposed to the Raman regime. The density modulation is therefore

$$\begin{aligned} \delta h^{(l-M)} = & (-ih^{(0)}\omega/\nu^2 w_0^4) \\ & \times (u^{(-M)}\,\delta E_r^{(l)} + v^{(-M)}\,\delta E_\theta^{(l)}). \end{aligned} \tag{41}$$

We now turn to Maxwell's equations. For $l^* = l$, the source in Maxwell's equation [Eq. (22)] is, to lowest order,

$$\begin{aligned} S_\pm^{(l)} = & i\omega(u^{(M)} \pm iv^{(M)})\delta h^{(l-M)} \\ & + i\omega h^{(0)}(\delta u^{(l)} \pm i\delta v^{(l)}). \end{aligned} \tag{42}$$

It is easy to show that the dominant term on the rhs of Eq. (42) is the first term, which results from the density modulation. The source $S_\pm^{(l)}$ is $O(\epsilon^{3-2p_q})$ and following assumption (40), $S_\pm^{(l)}$ is $o(1)$. Finally, the Maxwell equations become

$$\begin{aligned} & \left\{ r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \left[\omega^2 - \left(\omega\frac{\gamma_0}{w_0} - kM\right)^2 \right.\right. \\ & \quad \left.\left. - 2\nu\left(\omega\frac{\gamma_0}{w_0} - kM\right) - \nu^2 - \frac{l}{r^2}(l \pm 2)\right]\right\} \\ & \quad \times (r\,\delta E_r^{(l)} \pm ir\,\delta E_\theta^{(l)}) \\ & \quad = \frac{h^{(0)}\omega^2}{\nu^2 w_0^4}(u^{(M)} \pm iv^{(M)}) \\ & \qquad \times (u^{(-M)}\,\delta E_r^{(l)} + v^{(-M)}\,\delta E_\theta^{(l)}). \end{aligned} \tag{43}$$

Two boundary conditions are the regularity of $\delta E_r^{(l)}$ and $\delta E_\theta^{(l)}$ at the origin. Two additional boundary conditions are

$$\frac{\partial}{\partial r}(r\delta E_r^{(l)})_{r=R} = \delta E_\theta^{(l)}(R) = 0 \tag{44}$$

and they follow the assumption that a perfectly conducting wall is located at $r = R$.

The two second-order ordinary differential equations with the two-point boundary conditions are solved for the eigenfunctions $\delta E_r^{(l)}$ and $\delta E_\theta^{(l)}$ and for the eigenvalue ν. To simplify the notation, we omit the superscript l in what follows, with the understanding that the perturbed fields comprise, to lowest order, the l helical harmonic.

The dynamics of the beam is contained in the current terms on the rhs of Eqs. (43). To lowest order, the density modulation yields the dominant contribution to the perpendicular current. As in the 1-D analysis this density modulation is expressed, to lowest order, as a function of the wave fields and not as a solution of differential equations. Thus the final truncated set of differential equations (43) comprise a fourth-order system, which is the order of the Maxwell equations.

In Sec. IV we derive a dispersion relation for the eigenvalues.

IV. NONREAL EIGENVALUES

Since we are interested in nonreal eigenvalues, which correspond to unstable modes, we start by determining the domains in the complex ν plane, where nonreal eigenvalues can be found. We multiply Eqs. (43) by $\delta E_r^* \mp i\delta E_\theta^*$, integrate by parts, and using the boundary conditions (44) we obtain

$$(-2\nu d-\nu^2)p_1=p_4+p_3/\nu^2, \tag{45}$$

where

$$\begin{aligned} d&=[\omega(\gamma_0/w_0)-kM],\\ p_1&=\int_0^R dr\, r(|\delta E_r|^2+|\delta E_\theta|^2),\\ p_2&=|\delta E_r|^2_{r=0}+|\delta E_\theta|^2_{r=0}\\ &\quad+2\int_0^R\frac{dr}{r}\left(\left|\frac{\partial}{\partial r}(r\delta E_r)\right|^2+\left|\frac{\partial}{\partial r}(r\delta E_\theta)\right|^2\right)\\ &\quad+l^2\int_0^R\frac{dr}{r}(|\delta E_r|^2+|\delta E_\theta|^2), \qquad (46)\\ p_3&=\frac{\omega^2}{w_0^4}\int_0^R dr\, h^{(0)}|u^{(M)}\delta E_r+v^{(-M)}\delta E_\theta|^2,\\ p_4&=p_2-(\omega^2-d^2)p_1. \end{aligned}$$

The quantities p_1, p_2, and p_3 are real and positive. Looking for nonreal ν, we take the imaginary part of Eq. (45) and obtain

$$(d+\nu_r)p_1=(p_3/|\nu|^4)\nu_r. \tag{47}$$

Since $\omega \gg 1$, d is also much larger than 1. Using the Schwarz inequality we conclude that

$$\frac{(d+\nu_r)|\nu|^4}{\nu_r}=\frac{p_3}{p_1}<F\equiv\frac{\omega^2\gamma_0^2}{w_0^4}\left(1-\frac{w_0^2}{\gamma^{(0)2}}\right)\times h^{(0)}_{\max}(|u^{(M)}|^2_{\max}+|v^{(M)}|^2_{\max}). \tag{48}$$

The domain in the complex ν plane, where nonreal eigenvalues are allowed, is shown schematically in Fig. 1. Since $F \ll 1$, we have

$$x_1\cong(F/d)^{1/3},\quad x_2\cong(i3^{1/2}/2^{4/3})(F/d)^{1/3},\quad x_3\cong-d-F/d^3. \tag{49}$$

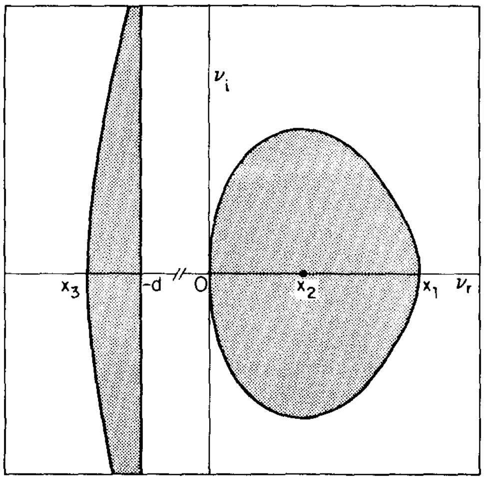

FIG. 1. The domains in the complex ν plane where nonreal eigenvalues are allowed.

That part of the domain where $\nu_r < -d$ corresponds to a negative $(q-lk)$. These waves are backward propagating. Our model is not valid for modes with eigenvalues of such order of magnitude and thus we do not discuss them here.

We now turn to that part of the domain in the half-plane of positive real part. We divide the eigenvalues here into two groups. The first group includes eigenvalues large enough so that the rhs of Eqs. (43) is $o(1)$. For this case Eqs. (43) can be solved perturbatively. The second group consists of smaller eigenvalues where the rhs of Eqs. (43) is $O(1)$ or larger. The first group consists of only one pair of eigenvalues, complex conjugates of each other. They have the largest imaginary part in that domain and are calculated correctly by our model. We obtain their value in a closed form. The second group consists of an infinite number of smaller eigenvalues with an accumulation point at the origin of the complex ν plane.

We now calculate the eigenvalues of the first group, for which the rhs of Eqs. (43) is $o(1)$. In our model, the rhs of this equation has to be $o(1)$. Thus all the eigenvalues which are calculated correctly by our model belong to this group.

To zeroth order we neglect the terms that are $o(1)$. These are the rhs of the equations and the terms which involve ν, since $\nu_0 = 0$. We obtain the vacuum equations

$$\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)+\left(\beta^2-\frac{l(l\pm 2)}{r^2}\right)\right](r\delta E_r\pm ir\delta E_\theta)=0, \tag{50}$$

with the same boundary conditions as before, and

$$\beta^2\equiv\omega^2-[\omega(\gamma_0/w_0)-kM]^2. \tag{51}$$

The solutions of these equations are well known. These are the TE modes

$$\delta E_r=(l/\beta r)J_l(\beta r), \tag{52a}$$

$$\delta E_\theta=i[J_l(\beta r)]_{,\beta r}, \tag{52b}$$

where

$$\beta=\alpha/R \tag{52c}$$

and α is one of the roots of

$$J_l'(\alpha)=0 \quad (52d)$$

and the TM modes

$$\delta E_r=[J_l(\beta r)]_{,\beta r}, \quad (53a)$$

$$\delta E_\theta=ilJ_l(\beta r)/\beta r, \quad (53b)$$

where again $\beta=\alpha/R$ and α is one of the roots of

$$J_l(\alpha)=0. \quad (53c)$$

From definitions (51) and (52c) and conditions (52d) and (53c), we see that our model holds for two infinite sets of resonant frequencies ω_r which obey

$$\alpha^2/R^2=\omega_r^2-[\omega_r(\gamma_0/w_0)-kM]^2 \quad (54)$$

and where α takes the infinite set of roots of (52d) or (53c). The frequency of the wave has to obey condition (54) for the eigenvalue to have a large imaginary part. This is the resonance condition.

For each α there are two frequencies which obey Eq. (54). The smaller of them is $O(1)$. However, since we limited ourselves to high frequencies [Eq. (33)], our model is valid for only the larger of the two frequencies, which is

$$\omega_r=kM\gamma_0w_0+[k^2M^2w_0^4-(\alpha/R)^2]^{1/2}. \quad (55)$$

This is the well-known relation for FEL's. We assume α/R is $O(1)$ and approximate ω_r as

$$\omega_r=kMw_0(\gamma_0+w_0). \quad (56)$$

We now turn to the first-order equations. We expand the frequency ω and the eigenvalue ν as follows:

$$\omega=\omega_r+\omega_1, \quad \nu=\nu_1, \quad (57)$$

where

$$\omega_1\ll\omega_r, \quad \nu_1\ll 1. \quad (58)$$

The first-order form of Eqs. (43) is

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\,\delta E_{r,1}\pm ir\,\delta E_{\theta,1})\right)+\left[\left(\frac{\alpha}{R}\right)^2-\frac{l(l\pm 2)}{r^2}\right](r\,\delta E_{r,1}\pm ir\,\delta E_{\theta,1})=f_\pm$$
$$=2(kM\omega_1+\omega_r\nu_1)(r\,\delta E_{r,0}\pm ir\,\delta E_{\theta,0})+\frac{h^{(0)}}{\nu_1^2}\frac{\omega_r^2}{w_0^4}(u^{(M)}\pm iv^{(M)})\times(u^{(-M)}\,\delta E_{r,0}+v^{(-M)}\,\delta E_{\theta,0}), \quad (59)$$

where

$$\omega_1=O(\epsilon^{p_q-1}). \quad (60)$$

In Eq. (59) $\delta E_{r,0}$ and $\delta E_{\theta,0}$ are the vacuum solutions given by (52a) and (52b) or (53a) and (53b) and α is given by (52d) or (53c).

Equation (59) is an inhomogeneous boundary value equation for x_+ and x_-:

$$x_\pm\equiv r\,\delta E_{r,1}^{(l)}\pm ir\,\delta E_{\theta,1}^{(l)}. \quad (61)$$

The homogeneous equations (50) have regular solutions $g_\pm$ and irregular solutions $y_\pm$ of the form

$$g_\pm=rJ_{l\pm 1}(\beta r), \quad (62a)$$

$$y_\pm=rY_{l\pm 1}(\beta r), \quad (62b)$$

and β is given by (52c). The general regular solutions of Eq. (59) are

$$x_+=\frac{\pi}{2}y_+\int_0^r\frac{g_+f_+}{r'}dr'-\frac{\pi}{2}g_+\int_0^r\frac{y_+f_+}{r'}dr'+Cg_+, \quad (63a)$$

$$x_-=\frac{\pi}{2}y_-\int_0^r\frac{g_-f_-}{r'}dr'-\frac{\pi}{2}g_-\int_0^r\frac{y_-f_-}{r'}dr'+Cg_-. \quad (63b)$$

The functions f_+ and f_- on the rhs contain δE_r and δE_θ, which are given by Eqs. (52) or (53). The solutions have to obey the boundary conditions at the wall:

$$x_+(R)-x_-(R)=0, \quad (64a)$$

$$x_+'(R)+x_-'(R)=0. \quad (64b)$$

Since g_+ and g_- also obey the boundary conditions (60), we obtain

$$y_+(R)\int_0^R\frac{dr}{r}g_+f_+-y_-(R)\int_0^R\frac{dr}{r}g_-f_--g_+(R)\times\int_0^R\frac{dr}{r}(y_+f_+-y_-f_-)=0, \quad (65a)$$

$$y_+'(R)\int_0^R\frac{dr}{r}g_+f_++y_-'(R)\int_0^R\frac{dr}{r}g_-f_--g_+'(R)\times\int_0^R\frac{dr}{r}(y_+f_+-y_-f_-)=0. \quad (65b)$$

We now multiply Eq. (65a) by $g_+'(R)$ and subtract from it the product of Eq. (65b) by $g_+(R)$. We use the Wronskian relations between g and y and Eq. (64) and obtain the solvability condition

$$\int_0^R\frac{dr}{r}(g_+f_++g_-f_-)=0. \quad (66)$$

The solvability condition is an equation for the eigenvalue ν_1. Once we have found ν_1, f_+ and f_- are known and the eigenfunctions x_+ and x_- are given by Eq. (63). The constant C in Eq. (63) is not determined so far. If needed, we may determine it by requiring that the eigenfunction be normalized to unity.

The solvability condition (66) is a dispersion relation, which we may write explicitly as

$$2(kM\omega_1+\omega_r\nu_1)\int_0^R dr\,r(|\delta E_r|^2+|\delta E_\theta|^2)+\frac{\omega_r^2}{w_0^4\nu_1^2}\int_0^R dr\,h^{(0)}[|\delta E_r|^2u^{(M)^2}+|\delta E_\theta|^2v^{(M)^2}+2u^{(M)}v^{(M)}i\,\mathrm{Im}(\delta E_r\,\delta E_\theta^*)]=0. \quad (67)$$

Here δE_r and δE_θ are given by Eqs. (52) or (53). It might be useful to write the dispersion relation in the following compact form:

$$(\xi+\lambda)\lambda^2=-PR_1/R_2, \quad (68)$$

where

$$\xi\equiv 2kM\omega_1 \quad (69a)$$

and is the mismatch parameter,

$$P\equiv 4\omega_r^4/w_0^4, \quad (69b)$$

$$R_1 \equiv \int_0^R dr\, h^{(0)} [\,|\delta E_r|^2 u^{(M)^2} + |\delta E_\theta|^2 v^{(M)^2} + 2u^{(M)} v^{(M)} i\, \mathrm{Im}(\delta E_r\, \delta E_\theta^*)\,], \quad (69c)$$

$$R_2 \equiv \int_0^R dr\, r(|\delta E_r|^2 + |\delta E_\theta|^2), \quad (69d)$$

and the normalized eigenvalue is

$$\lambda \equiv 2\omega_r \nu_1. \quad (69e)$$

The dispersion relation (68) is a cubic polynomial in the eigenvalue λ of a form similar to that of the 1-D theories of the free electron laser in the strong-pump regime.[4] There are two nonreal eigenvalues, complex conjugates of each other, with imaginary part (for $\xi = 0$):

$$|\mathrm{Im}\,\lambda| = (\sqrt{3}/2)(PR_1/R_2)^{1/3}. \quad (70)$$

The 3-D effects of the wiggler field, the finite beam thickness, and the waveguide are expressed through R_1 and R_2 [Eq. (69)]. From both Eqs. (68) and (70) we conclude that

$$p_q = \tfrac{4}{3}. \quad (71)$$

Inequality (40) is indeed satisfied. Again, this is also the basic scaling in the 1-D theories of the free electron laser in the strong-pump regime.

From Eqs. (60) and (33) we find also that

$$\omega_1/\omega = O(\epsilon^{p_q}) = O(\epsilon^{4/3}). \quad (72)$$

In addition we note that since, to lowest order, the wave fields are the waveguide modes, optical guiding is absent in the case we treat here.

We now turn to the second group of eigenvalues. The eigenvalues of the second group are smaller than those of the first group. It is possible to show that $\nu = 0$ is an accumulation point. However, we do not study this group in detail both because our model is not a good approximation for them, and because the mode of the first group is the most unstable and is dominant. Instead we find the values of these eigenvalues for a simple particular case.

Let us look at the case

$$kR \ll 1 \quad (73)$$

and apply the paraxial approximation. From Eq. (4) we obtain

$$v^{(M)} = iu^{(M)} = iu^{(-M)} = -v^{(-M)} = iBr^{M-1}, \quad (74)$$

where B is a positive constant. Equations (43) become

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}x_+\right) + \left(\omega^2 - d^2 - 2\nu d - \nu^2 - \frac{l}{r^2}(l+2)\right)x_+ = 0, \quad (75a)$$

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}x_-\right) + \left(\omega^2 - d^2 - 2\nu d - \nu^2 - \frac{l}{r^2}(l-2) - \frac{2h^{(0)}}{\nu^2}B^2 r^{2M-3}\frac{\omega^2}{w_0^4}\right)x_- = 0. \quad (75b)$$

The solution of Eq. (75a) is

$$x_+ = D_+ r J_{l+1}(\beta_+ r), \quad (76a)$$

where

$$\beta_+^2 = \omega^2 - d^2 - 2\nu d - \nu^2. \quad (76b)$$

In order to simplify the solution of Eq. (75b), we choose

$$h^{(0)} = \hat{h} r^{3-2M}, \quad (77)$$

which gives

$$x_- = D_- r J_{l-1}(\beta_- r), \quad (78a)$$

$$\beta_-^2 = \omega^2 - d^2 - 2\nu d - \nu^2 - 2\hat{h}B^2\omega^2/\nu^2 w_0^4. \quad (78b)$$

By using the boundary conditions (64) we obtain

$$D_+ J_{l+1}(\beta_+ R) - D_- J_{l-1}(\beta_- R) = 0, \quad (79a)$$

$$D_+[rJ_{l+1}(\beta_+ R)]_{,r} + D_-[rJ_{l-1}(\beta_- R)]_{,r} = 0. \quad (79b)$$

Nontrivial solutions for D_+ and D_- exist if the following dispersion relation is satisfied:

$$\beta_+ J_l(\beta_+ R) J_{l-1}(\beta_- R) - \beta_- J_l(\beta_- R) J_{l+1}(\beta_+ R) = 0. \quad (80)$$

For small ν,

$$\beta_+^2 = \omega^2 - d^2, \quad (81a)$$

$$\beta_-^2 = \omega^2 - d^2 - 2\hat{h}B^2\omega^2/\nu^2 w_0^4. \quad (81b)$$

It is easy to show that $\nu = 0$ is a limit point of the roots of Eq. (80). For given ω and β_+ we look for the roots $\beta_- R$ of

$$\frac{\beta_- R J_l(\beta_- R)}{J_{l-1}(\beta_- R)} = \frac{\beta_+ R J_l(\beta_+ R)}{J_{l+1}(\beta_+ R)}. \quad (82)$$

From the asymptotic form of the Bessel functions it is clear that for large $\beta_- R$ the lhs is a periodic function of $\beta_- R$ with 2π periodicity, and its range is $[-\infty, \infty]$. In every period there are two roots $\beta_- R$ for Eq. (82), with a corresponding series ν_n^2, where, with (81b),

$$\nu_n^2 \cong -2\hat{h}B^2\omega^2/\beta_{-n}^2 w_0^4 \to 0_-. \quad (83)$$

Thus near the limit point the eigenvalues ν_n are pure imaginary.

V. CONCLUSIONS

We presented a theory of a thick beam free electron laser, operating in the strong-pump regime, which included 3-D effects of the wiggler field, the waveguide, and the beam thickness. The wiggler field was of a general magnetic multipole number and self-fields of the beam were considered in the equilibrium. We assumed the thickness of the beam to be comparable to the wiggler wavelength and to the waveguide radius. The result of this assumption was that in order to operate in the strong-pump regime, there were constraints on the beam density and energy and on the wave frequency. These constraints could be relaxed easily if the beam thickness were smaller than the wiggler wavelength. The relaxation of these constraints while the thickness of the beam remains unchanged would be even more attractive. The case of a high-current thick beam FEL, operating in the Raman regime, is under investigation.

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