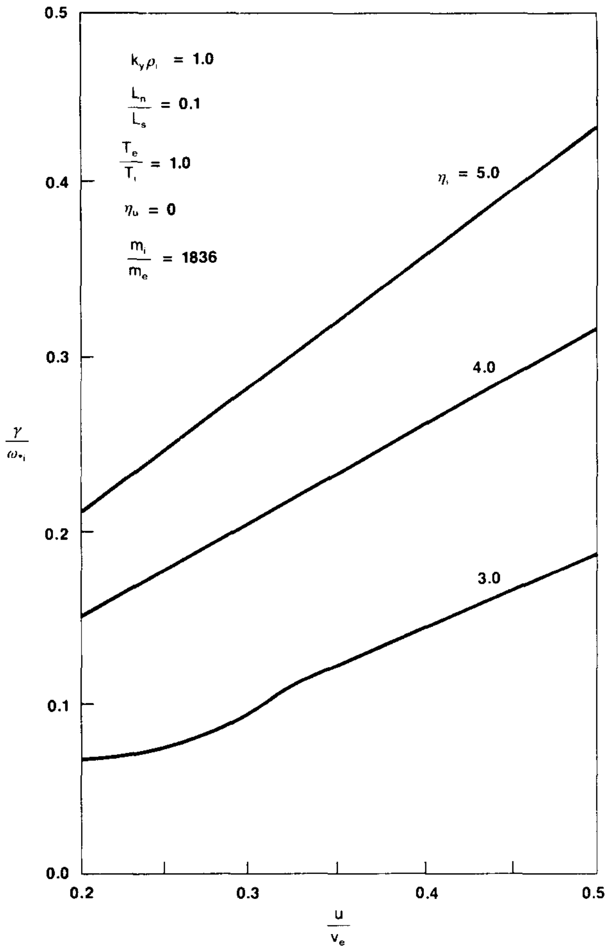

FIG. 4. Dependence of current-coupled η_i-mode growth rate on ion temperature gradient for three values of the current parameter.

destabilizing and mode altering influence of field-aligned current as discussed in this paper. We should note, however, that the transport consequences of the destabilizing effects of current may not be as significant as the coupling to the resistive g mode. If we estimate the anomalous transport with the intuitive $D \sim (\Delta x)^2/\Delta t \sim \gamma/k^2$, and note that the current destabilization increases growth by a factor of order 2 while shifting maximum growth from $k_y \rho_i \lesssim 0.1$ to $k_y \rho_i \gtrsim 1.0$, then we would expect a 50-fold decrease in the rate of anomalous diffusion. This is encouraging, but clearly, a more comprehensive nonlinear treatment-coupled η_i mode is required to elucidate the transport consequences.

Other possibly important effects characteristic of RFP's that are missing from both of these theories are beta effects and effects associated with toroidal geometry. It appears that the accurate assessment of the role of the η_i mode in RFP's awaits a comprehensive linear kinetic theory that includes all of the sources of free energy—field curvature, diamagnetic, and field-aligned currents—and which is also capable of dealing with the finite beta, collisionality, and strong magnetic shear typical of those devices.

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Absorption of waves at the ion-cyclotron frequency range by drifting electrons

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Waves in the ion-cyclotron range of frequencies are shown to be Landau damped by drifting electrons in the presence of an equilibrium electric field. Absorption occurs even if both the electric component of the wave parallel to the equilibrium magnetic field and the component of the wave vector perpendicular to the equilibrium magnetic field are zero.

Absorption of waves on the order of the ion-cyclotron frequency by electrons has been observed recently in the Elmo Bumpy Torus-S[1] and in the Tandem Mirror Experiment-U.[2] In both experiments it seemed that the resonance condition for Landau damping, $\omega - k_z v_t \cong 0$, was satisfied, where ω is the wave frequency, k_z is the wavenumber in the direction of the equilibrium magnetic field, and v_t is the thermal velocity of the heated electrons. Waves whose electric field parallel component E_z is finite are expected to be heavily damped indeed. However, the component E_z of waves at the ion-cyclotron frequency range is usually very small.[3] Even if E_z is zero there can be an absorption attributable to a finite Larmor radius effect if the component of the wave vector perpendicular to the equilibrium magnetic field k_x is not zero. This effect, called transit time damping, was described by Stix[4] and later by Scharer *et al.*[5] The purpose of this Brief Communication is to show that even when both E_z and k_x are zero, there can be a substantial absorption of the waves by the electrons resulting from the presence of an equilibrium electric field accompanied by a drift of the electrons. Such equilibrium electric fields have been observed in both EBT-S and TMX-U. If the drift velocity of the electrons is v_d, and the resonance condition for Landau damping holds, the quantity Q, which describes the rate of dissipation, is about $(\omega^2/\omega_p^2)(v_t^2/v_d^2)$, where ω_p is the electron plasma frequency. For values of the parameters such as those in fusion plasmas, Q is much smaller than 1. This corresponds to strong damping.

We study a simple model problem of given waves propagating in an infinite uniform plasma. The current is obtained by solving the Vlasov equation for the electrons, and the product of this current and the wave electric field yields the rate of electron heating by the wave. A similar analysis for a case of nonuniform plasma was studied recently.[6] The present analysis is restricted to uniform plasma only but allows an equilibrium electrostatic field and the corresponding steady-state drift velocity.

Let us consider an infinite uniform plasma immersed in uniform static electric field $\mathbf{E}_0$ and magnetic field $\mathbf{B}_0$ of the form

$$\mathbf{E}_0 = \hat{x}E_0, \quad \mathbf{B}_0 = \hat{z}B_0. \tag{1}$$

The equilibrium distribution function of the electrons G is a function of the constant of motion p only,

$$p \equiv v_x^2 + (v_y + v_d)^2 + v_z^2, \quad v_d \equiv cE_0/B_0, \tag{2}$$

where $\mathbf{v}$ is the electron velocity. Waves whose electric component $\mathbf{E}_1$ and magnetic component $\mathbf{B}_1$ are

$$\mathbf{E}_1 = \mathbf{E}e^{i(k_x x + k_z z - \omega t)}, \quad \mathbf{B}_1 = \mathbf{B}e^{i(k_x x + k_z z - \omega t)} \tag{3}$$

propagate in the plasma. The perturbed electron distribution function is

$$f = G(p) + F(\mathbf{v})e^{i(k_x x + k_z z - \omega t)}, \tag{4}$$

where F obeys the linearized Vlasov equation,

$$i(-\omega + k_z v_z + k_x v_x)F - eE_0\frac{\partial F}{\partial v_x} - \Omega\left(v_y\frac{\partial F}{\partial v_x} - v_x\frac{\partial F}{\partial v_y}\right) = S \equiv \frac{dp}{dt}\frac{dG}{dp} = \frac{2e}{m}[-\mathbf{E}\cdot v + v_d(-E_y + v_x B_z - v_z B_x)]\frac{dG}{dp}. \tag{5}$$

Here Ω is the eB_0/mc, e and m are the electron charge and mass, and c is the velocity of light in vacuum. By using one of Maxwell's equations we have

$$B_x = -(k_z c/\omega)E_y. \tag{6}$$

The unique periodic solution of the Vlasov equation is[6]

$$F(t) = \{\exp[2\pi i(-\omega + k_z v_z)/\Omega] - 1\}^{-1} \times \int_0^{t_0} dt'\, S(t+t')\exp\left(i(-\omega + k_z v_z)t' + \frac{ik_x}{\Omega}[v_y(t+t') - v_y(t)]\right), \tag{7}$$

where the integration is performed along the characteristics:

$$v_x(t+t') = [v_y(t) + v_d]\sin\Omega t' + v_x(t)\cos\Omega t',$$
$$v_y(t+t') = -v_d + [v_y(t) + v_d] \times \cos\Omega t' - v_x(t)\sin\Omega t', \tag{8}$$

and $t_0 = 2\pi/\Omega$. We restrict ourselves now to the case $|(-\omega + k_z v_z/\Omega| \ll 1$. This applies in particular when the frequency ω is on the order of Ω_i, the ion-cyclotron frequency. Then if $k_z v_z \sim \omega$, the quantity $(\omega - k_z v_z)/\Omega$ is on the order of m/m_i, where m_i is the ion mass. Thus we approximate F as

$$F(t) = [-(-\omega + kv_z)t_0]^{-1}\int_0^{t_0} dt'\, S(t+t') \times \exp\left(\frac{ik_x}{\Omega}[v_y(t+t') - v_y(t)]\right). \tag{9}$$

Let us consider the terms in the last integral that are proportional to

$$I = \int_0^{t_0} dt'\, v_x(t+t')\exp\left(\frac{ik_x}{\Omega}v_y(t+t')\right). \tag{10}$$

Since we may write v_x and v_y in the following form:

$$v_x = -v_\perp \sin\Omega t'', \quad v_y = -v_d + v_\perp\cos\Omega t'', \quad v_\perp^2 \equiv v_x^2 + (v_y + v_d)^2, \tag{11}$$

the integral (10) is zero. We perform the integration in the expression for F [Eq. (9)] by using Eq. (11), and finally obtain for F

$$F = \frac{i}{(-\omega + k_z v_z)}\exp\left(-\frac{ik_x}{\Omega}(v_d + v_y)\right) \times \left[v_z E_z J_0 + E_y\left(iv_\perp J_1 - \frac{k_z v_z}{\omega}v_d J_0\right)\right]\frac{2e}{m}\frac{dG}{dp}. \tag{12}$$

Here J_n is the Bessel function of order n and its argument is $(k_x v_\perp/\Omega)$.

Since we have found the perturbed distribution function we may now calculate the current $\mathbf{j} = -e\int \mathbf{v}F\,d^3v$. The component j_x is zero because the x component of the integrand is antisymmetric with respect to v_x. We now specify the form of G,

$$G = (N_0/\pi^{3/2}v_t^3)\exp(-p/v_t^2), \tag{13}$$

and by performing the integration we obtain

$$j_z = (i\omega_p^2/\pi\omega)\rho[1 + \rho Z(\rho)]\{\rho h_0(r)E_z + [ih_1(r) - sh_0(r)]E_y\},$$
$$j_y = (i\omega_p^2/\pi\omega)\{-\rho[1 + \rho Z(\rho)][sh_0(r) + ih_1(r)]E_z + [(1 + \rho Z(\rho))s^2 h_0(r) + ish_1(r) + \rho Z(\rho)h_2(r)]E_y\}, \tag{14}$$

where $\omega_p^2 \equiv 4\pi N_0 e^2/m$, $\rho = \omega/k_z v_t$, $s = v_d/v_t$, $s = v_d/v_t$, $r = k_x v_t/\Omega$, and Z is the plasma dispersion function. The functions $h_0(r)$, $h_1(r)$, and $h_2(r)$, given as

$$h_0(r) \equiv \int_0^\infty dt\, t J_0^2(rt) e^{-t^2},$$
$$h_1(r) \equiv \int_0^\infty dt\, t^2 J_0(rt) J_1(rt) e^{-t^2}, \tag{15}$$
$$h_2(r) \equiv \int_0^\infty dt\, t^3 J_1^2(rt) e^{-t^2},$$

can be formally represented by hypergeometric functions. If the component E_z of the waves is of the same order of magnitude as E_y, then, since r and s are usually small, the current to lowest order is

$$j_z = (i\omega_p^2/2\pi\omega)\rho^2[1 + \rho Z(\rho)]E_z, \quad j_y = 0. \tag{16}$$

The rate of electron heating is

$$\mathrm{Re}(\mathbf{E}^*\cdot\mathbf{j}) = \frac{\omega_p^2}{2\pi\omega}\rho^3\, \mathrm{Im}[Z(\rho)]|E_z|^2 = \frac{\omega_p^2}{\sqrt{\pi}\omega}\rho^3 e^{-\rho^2}|E_z|^2. \tag{17}$$

The value of Q expresses the rate of dissipation

$$Q \equiv (\omega \mathbf{E}^*\cdot\mathbf{E})/\mathrm{Re}(\mathbf{E}^*\cdot\mathbf{j}). \tag{18}$$

In this case Q is

$$Q = \left[2\sqrt{\pi}\omega^2/(\omega_p^2\, \rho^3 e^{-\rho^2})\right](\mathbf{E}^*\cdot\mathbf{E}/E_z^* E_z), \tag{19}$$

and is very small, corresponding to heavy damping. However, waves at the ion-cyclotron frequency range have very small E_z as a result of the high conductivity of the electrons parallel to the equilibrium magnetic field. Taking E_z to be zero, we calculate next the work done by the component E_y. When there is no drift, the heating is a finite Larmor radius effect that requires a finite k_x. The rate of heating is then given by

$$\mathrm{Re}(E_y^* j_y) = (\omega_p^2/\sqrt{\pi}\omega)\rho e^{-\rho^2} h_2(r)|E_y|^2, \tag{20}$$

in agreement with Refs. 4 and 6. If there is a drift, then even if the propagation is parallel and k_x is zero, there is heating given by

$$\mathrm{Re}(E_y^* j_y) = (\omega_p^2/\sqrt{\pi}\omega)\rho e^{-\rho^2} s^2 |E_y|^2. \tag{21}$$

One may wonder why the presence of a drift, which may be eliminated by a Lorentz transformation, can cause heating. Assume for simplicity that k_x is zero and let us solve the problem in a frame moving with a velocity of $\bar{v}_y = -v_d \ll c$. The equilibrium fields and distribution function in the moving frame are

$$\mathbf{E}_0' = 0, \quad \mathbf{B}_0' \cong B_0\hat{z}, \quad G = (N_0/\pi^{3/2}v_t^3)\exp(-p'/v_t^2),$$
$$p' = v_x'^2 + v_y'^2 + v_z'^2, \tag{22}$$

where the prime denotes quantities in the moving frame. The wave fields,which had a perpendicular electric component only in the laboratory frame, also have a parallel component in the moving frame. They are

$$\mathbf{E}' = [\hat{y} - (k_z v_d/\omega)\hat{z}]E_y e^{i(k_z z - \omega t)},$$
$$\mathbf{B}' = -(k_z c/\omega)\hat{x} E_y e^{i(k_z z - \omega t)}. \tag{23}$$

The heating attributable to the component E_z' can be found by substituting the value of E_z' into Eq. (17). The rate of heating agrees with that given by Eq. (20), which was derived in the laboratory frame.

For both cases, the case of oblique propagation and the case of drift, we can write the Q of the system as

$$Q = \pi^{1/2}(\omega/\omega_p)^2/(\rho e^{-\rho^2} d), \tag{24}$$

where, when s is zero, d is $h_2(r)$ ($r^2/4$ for r small), and when r is zero, d is s^2.

In fusion plasmas ω_p is typically on the order of Ω and since ω is on the order of Ω_i, ω^2/ω_p^2 is on the order of $(m/m_i)^2$. The number d in Eq. (24) is usually small, but in hot plasmas it can be much larger than the square of the mass ratio. As a result, near the Landau resonance ($\rho \simeq 1/\sqrt{2}$), Q is much smaller than 1, corresponding to strong absorption.

Having completed the analysis of the model problem, we would like now to estimate the values of some of the parameters that appear in the model and then suggest more realistic models for further study.

In cylindrical plasmas, the parameter r, characterizing the finite Larmor radius source of heating, is usually at least on the order of the ratio of the Larmor radius and the perpendicular scale length. If we assume that the electrostatic potential energy of particles at the edge of the cylinder relative to the axis is comparable to their mean kinetic energy, it is easy to show that s is also on the order of the ratio of the Larmor radius and the perpendicular scale length. Thus the finite Larmor radius effect and the drift are expected to have comparable contributions to the heating. In mirror devices the potential at the end cells may be high, which could cause the absorption attributable to the drifting electrons to be dominant.

It is important to evaluate the magnitude of E_z, since even a small E_z can cause strong heating [see Eq. (18)]. Here we give a simple argument to show that in the presence of drift, E_z can differ from its zero value. Assume a cold fluid with a drift velocity $\mathbf{v}_d = -v_d\,\hat{y}$ in the presence of the static fields given by Eq. (1). Since the pressure is zero, the parallel component of the linearized momentum equation is

$$m\frac{dv_{1z}}{dt} = -eE_z - e\frac{v_d}{c}B_x, \tag{25}$$

where v_{1z} is the perturbed electron velocity. In the absence of drift, the zero electron mass approximation results in $E_z = 0$. However, the presence of drift yields [with the use of Eq. (6)]

$$E_z/E_y = v_d k_z/\omega. \tag{26}$$

This ratio is on the order of the magnitude of s near the resonance. A parallel electric field of that magnitude can cause heating comparable to that heating described by Eq. (20).

We have studied here a simple model problem in an infinite uniform plasma in order to demonstrate the effect of drift on electron heating by low frequency waves. In order to investigate this phenomenon in a laboratory plasma a more complicated model is required, which will include the effects of the gradients of the plasma density and the static electric field and which will solve for the wave fields in a self-consistent manner.

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The progressive wave approach analyzing the decay of a sawtooth profile in magnetogasdynamics

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An asymptotic approach is used to analyze the main features of weakly nonlinear waves propagating through an electrically conducting gas permeated by a transverse magnetic field. The analysis leads to an evolution equation, which characterizes the wave process in the high-frequency domain. The growth equation for an acceleration front is recovered as a special case. The influence of the magnetic field on the decay behavior of a sawtooth profile, which is headed by a weak shock front and ends with a magnetosonic disturbance, is investigated. A remarkable difference between the plane and cylindrical wave profiles is noted; for instance, when the adiabatic index γ is 2, the field does not affect the decay behavior of plane waves but does affect cylindrical waves.

In the study of a physical phenomenon ruled by a quasilinear hyperbolic system of equations, it is theoretically possible to look for progressive wave solutions. A general discussion of small-amplitude nonlinear progressive waves has been given by Choquet-Bruhat,[1] who considered a shockless solution of hyperbolic partial differential equations that depend on a single phase function. Using the perturbation method devised by Choquet-Bruhat,[1] Germain,[2] Fusco,[3] and Fusco and Engelbrecht[4] analyzed nonlinear wave propagation in different material media, while Hunter and Keller[5] presented a method for finding a small-amplitude high-frequency wave solution of hyperbolic systems of quasilinear partial differential equations. In the present paper we use the same approach to analyze the decay behavior of a disturbance given in the form of a sawtooth profile that consists of a front shock at the right and a magnetosonic disturbance at the left. The propagation of a sawtooth profile that ends in a tail shock can be treated in the same way.

Assuming the electrical conductivity to be infinite, and the direction of the magnetic field orthogonal to the trajectories of fluid particles, the basic equations governing the fluid flow can be written as[6]

$$\rho_t + u\rho_x + \rho(u_x + mux^{-1}) = 0\,, \tag{1}$$

$$u_t + uu_x + \rho^{-1}(p_x + h_x) = 0\,, \tag{2}$$

$$p_t + up_x + \gamma p(u_x + mux^{-1}) = 0\,, \tag{3}$$

$$h_t + uh_x + 2h(u_x + mux^{-1}) = 0\,, \tag{4}$$

where u is the fluid velocity, p the pressure, ρ the density, γ the adiabatic index, $h = \mu H^2/2$ the magnetic pressure with H as the magnetic field strength, μ is the magnetic permeability, t the time, and x the spatial coordinate. Letter subscripts denote partial differentiation unless stated otherwise. The letter m takes values of 0 or 1 accordingly to whether the motion is planar or cylindrically symmetric, respectively.

Using matrix notation, Eq. (1)–(4) can be written as

$$U^i_t + A^{ij} U^j_x + B^i = 0\,, \quad i,j = 1,2,3,4, \tag{5}$$

where the U^i are components of a column vector $\mathbf{U}$ with components ρ, u, p, and h. The components A^{ij} of a 4×4 matrix $\mathbf{A}$ and B^i of a column vector $\mathbf{B}$ can be found by inspection of Eqs. (1)–(4).

The system (5) is a hyperbolic one with eigenvalues $u+c$, $u-c$, u, and u of the coefficient matrix $\mathbf{A}$. Here, $c = (a^2 + b^2)^{1/2}$ is the magnetosonic speed with $a = (\gamma p/\rho)^{1/2}$ as the speed of sound and $b = (2h/\rho)^{1/2}$ the Alfvén speed. The left and right eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue $u+c$ are

$$\mathbf{l} = (0, \rho c, 1, 1), \quad \mathbf{r}^T = (1, c/\rho, a^2, b^2)\,, \tag{6}$$

where a superscript means transposition.

We look for an asymptotic solution of Eq. (5) exhibiting the features of progressive waves. Let us assume the following asymptotic expansion:

$$U^i(x,t) = U^i_0 + \epsilon U^i_1(x,t,\xi) + O(\epsilon^2)\,, \tag{7}$$

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