Gain enhancement in a longitudinal magnetic wiggler by use of a coherently gyrophased electron beam

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The growth rate of the instability in the lowbitron [Appl. Phys. Lett. **39**, 845 (1981)]—the longitudinal wiggler beam interaction device—is shown to be enhanced significantly by the use of a coherently gyrophased electron beam.

I. INTRODUCTION

The lowbitron—a longitudinal wiggler beam interaction device—has been suggested by scientists at MIT[1–3] as an attractive source of intense submillimeter radiation. A thin pencil beam of relativistic electrons with large transverse velocity acquired before entering the interaction region travels on axis in combined uniform magnetic field and longitudinal periodic wiggler magnetic field. The equilibrium distribution function of the electrons is usually assumed to be randomly gyrophased and to depend only on $p_\perp$ and p_z, the perpendicular and parallel electron momenta. The use of a coherently gyrophased beam instead of the randomly gyrophased beam in the interaction with a uniform axial magnetic field was shown theoretically to increase the growth rate of the instability.[4,5] Recently a successful generation of such a beam using electrostatic fields was reported.[6] The device employing such a new type of beam was named the wiggler-free free electron laser. Our purpose in this paper is to show that the use of a coherently gyrophased beam substantially increases the gain also in the lowbitron.

In Sec. II we write the Maxwell–Vlasov equations for an electron beam of an arbitrary gyrophase distribution propagating along a parallel stratified magnetic field. In Sec. III we apply these equations to two cases. The first is a uniform magnetic field (the cyclotron maser instability and the wiggler-free free electron laser instability) and the second is the spatially periodic magnetic field (the lowbitron).

II. DERIVATION OF THE GOVERNING EQUATIONS

Consider an electron beam that propagates along an external static magnetic field of the approximated form

$$\mathbf{B}_0 = \frac{d\beta(z)}{dz}\hat{e}_z . \tag{1}$$

The electron beam is tenuous so that the static self-fields are negligible in comparison with the external magnetic field. The radial dependence of the various quantities in the system is assumed to be negligible also. The constants of motion of the electron are $p_\perp, p_z$, and $\phi - e\beta/cp_z$, where $\phi = tg^{-1}(p_y/p_x)$ is the gyrophase, $-e$ is the electron charge, and c is the velocity of light in vacuum. Upon entering the interaction region at $z=0$ the electron distribution function is denoted as $G(p_\perp,p_z,\phi)$. Along the interaction region the equilibrium distribution function f_0 is related to G via

$$f_0(p_\perp,p_z,\phi,z) = G\{p_\perp,p_z,\phi - (e/cp_z)[\beta(z)-\beta(0)]\} . \tag{2}$$

The electron beam interacts with an electromagnetic wave whose electric field $\mathbf{E}_1'$ and magnetic field $\mathbf{B}_1'$ are

$$\mathbf{E}_1' = \mathbf{E}(z)e^{-i\omega t}, \quad \mathbf{B}_1' = \mathbf{B}(z)e^{-i\omega t} . \tag{3}$$

The electron distribution function is approximately

$$f = f_0 + f_1(p_\perp,p_z,\phi,z)e^{-i\omega t} , \tag{4}$$

where f_1 is a solution of the linearized Vlasov equation

$$\left(-i\omega + v_z\frac{\partial}{\partial z} + \frac{e}{c\gamma m}\frac{d\beta}{dz}\frac{\partial}{\partial\phi}\right)f_1 = e\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)\cdot\frac{\partial f_0}{\partial\mathbf{p}} , \tag{5}$$

where m is the electron mass, $\gamma^2 = 1 + (p_\perp^2 + p_z^2/m^2c^2)$, and $\mathbf{v} = \mathbf{p}/m\gamma$. Fourier decomposing f_0 and f_1 with respect to the gyrophase ϕ,

$$f_0 = \sum_{n=-\infty}^{\infty} c_n(p_\perp,p_z,z)e^{in\phi}, \quad f_1 = \sum_{n=-\infty}^{\infty} A_n(p_\perp,p_z,z)e^{in\phi} , \tag{6}$$

and substituting these expressions in Eq. (5) we obtain the following equations for the coefficients A_n:

$$\begin{aligned}\left(-i\omega + v_z\frac{\partial}{\partial z} + \frac{ine}{c\gamma m}\frac{d\beta}{dz}\right)A_n = e\bigg[&\frac{i}{\sqrt{2}}\left(-E_+\frac{\partial c_{n-1}}{\partial p_\perp} + E_-\frac{\partial c_{n+1}}{\partial p_\perp}\right) + \frac{i}{\sqrt{2}p_\perp}[(n-1)E_+c_{n-1} + E_-(n+1)c_{n+1}]\\ &+ \frac{p_\perp}{\sqrt{2}\gamma cm}\left(B_+\frac{\partial c_{n-1}}{\partial p_z} + B_-\frac{\partial c_{n+1}}{\partial p_z}\right) + \frac{ip_z}{\sqrt{2}mc\gamma p_\perp}[-i(n-1)B_+c_{n-1} + i(n+1)B_-c_{n+1}]\\ &- \frac{p_z}{\sqrt{2}\gamma cm}\left(B_+\frac{\partial c_{n-1}}{\partial p_\perp} + B_-\frac{\partial c_{n+1}}{\partial p_\perp}\right) + E_z\frac{\partial c_n}{\partial p_z}\bigg] \equiv S_n .\end{aligned} \tag{7}$$

The complex components of a vector $\mathbf{F}$ are $F_\pm = (1/\sqrt{2}) \times (F_y + iF_x)$. If f_1 is zero at the entrance, the solutions of Eqs. (7) are

$$A_n = \int_0^z \frac{dz'}{v_z} \exp\left(\frac{i\omega\Delta z}{v_z} - in\Delta\chi\right) S_n \,, \tag{8}$$

where $\Delta z = z - z'$, $\Delta\chi = \chi(z) - \chi(z')$, and $\chi(z)$ is $e\beta/(cp_z)$. The perturbed current $\mathbf{J}$ and density ρ are determined by A_{+1}, A_{-1}, and A_0 only. The current components $J_\pm$ and the density ρ are

$$\begin{aligned} J_\pm &= \mp\sqrt{2}\, i\pi e \int_0^\infty p_\perp\, dp_\perp \int_{-\infty}^{\infty} dp_z\, v_\perp A_{\pm 1} \,, \\ \rho &= -2\pi e \int_0^\infty p_\perp\, dp_\perp \int_{-\infty}^{\infty} dp_z A_0 \,. \end{aligned} \tag{9}$$

At this stage we choose the equilibrium distribution function at the entrance G to be of the form

$$G = N_0/(2\pi p_\perp)\delta(p_\perp - p_{\perp 0})\delta(p_z - p_{z0})g(\phi) \,. \tag{10}$$

The constant N_0 is the equilibrium electron density. The perpendicular and parallel momenta of all the electrons are of the same magnitude, but the gyrophase angle is arbitrary. By using Eqs. (2) and (6) we obtain the following relations between the Fourier components of f_0 and g:

$$\begin{aligned} c_n &= N_0/(2\pi p_\perp)\exp[-in\chi(z)] \\ &\quad \times \delta(p_\perp - p_{\perp 0})\delta(p_z - p_{z0}) g_n \,, \end{aligned} \tag{11}$$

where $g(\phi) = \Sigma_{n=-\infty}^{\infty} g_n \exp(in\phi)$. With the above form of the coefficients c_n, the integrations in the momenta space in Eq. (9) are performed in a straightforward manner. The currents and the density are thus

$$-4\pi e J_\pm = \mp e^{\pm i\chi} \int_0^z dz' e^{i\omega\Delta z/v_z}(Q_\pm + S_\pm \Delta z + T_\pm \Delta\chi) \,, \quad -4\pi e\rho = \int_0^z dz'\, e^{i\omega\Delta z/v_z}(Q_0 + S_0\Delta z) \,, \tag{12}$$

where

$$\begin{aligned} Q_\pm &= \frac{i\omega_p^2}{\gamma}\left(-\frac{v_\perp}{\sqrt{2}v_z^2} a_z g_{\pm 1} - \frac{v_\perp^2}{2v_z^2}(b_+ g_{\substack{0\\-2}} + b_- g_{\substack{2\\0}}) + \frac{1}{v_z}(\mp i a_\pm - v_z b_\pm) g_0\right), \\ S_\pm &= -\frac{\omega\omega_p^2}{\gamma}\left(-\frac{v_\perp^2}{2v_z^3}(b_+ g_{\substack{0\\-2}} + b_- g_{\substack{2\\0}}) + \frac{v_\perp}{\sqrt{2}v_z}\left(1 - \frac{1}{v_z^2}\right) a_z g_{\pm 1} + \frac{iv_\perp^2}{2v_z^3}(-a_+ g_{\substack{0\\-2}} + a_- g_{\substack{2\\0}})\right), \\ T_\pm &= \mp\frac{\omega_p^2}{\gamma}\frac{v_\perp}{v_z^2}\left(\frac{v_\perp}{2}(b_+ g_{\substack{0\\-2}} + b_- g_{\substack{2\\0}}) + \frac{a_z}{\sqrt{2}} g_{\pm 1}\right), \\ Q_0 &= \frac{\omega_p^2}{\gamma}\left[\frac{v_\perp}{v_z}\left(\frac{i}{\sqrt{2}}(-a_+ g_{-1} + a_- g_1) - \frac{1}{\sqrt{2}v_z}(b_+ g_{-1} + b_- g_1)\right) + \left(1 - \frac{1}{v_z^2}\right) a_z g_0\right], \quad S_0 = (i\omega/v_z)Q_0 \,, \end{aligned} \tag{13}$$

and the zero subscripts have been omitted from $p_{\perp 0}$, p_{z0}, $v_{\perp 0}$, v_{z0}, γ_0, and χ_0. In the definitions (13) we redefine $\mathbf{v}$ as the ratio of the electron velocity to c, and define $\omega_p^2 = 4\pi N_0 e^2/(mc^2)$. Other new notations are

$$a_\pm = E_\pm e^{\pm i\chi}, \quad b_\pm = B_\pm e^{\pm i\chi}, \quad a_z = E_z \,. \tag{14}$$

Substituting the expressions (12) into the Maxwell equations we obtain a set of integrodifferential equations for $a_\pm$, a_z, and $b_\pm$. However, upon defining the new quantities

$$\begin{aligned} I_1^l &= \int_0^z dz' e^{i\omega\Delta z/v_z} S_l \,, \\ I_2^l &= \int_0^z dz' e^{i\omega\Delta z/v_z}[Q_l - z'S_l - \chi(z')T_l], \quad T_0 = 0 \,, \\ I_3^\pm &= \int_0^z dz' e^{i\omega\Delta z/v_z} T_\pm \,, \\ l &= 0, \pm \,, \end{aligned} \tag{15}$$

the integrodifferential equations become the following system of thirteen first-order ordinary differential equations:

$$\begin{aligned} \frac{db_\pm}{dz} &= \pm i\frac{d\chi}{dz} b_\pm \pm \omega a_\pm \mp i(I_2^\pm + zI_1^\pm + \chi I_3^\pm) \,, \\ \frac{da_\pm}{dz} &= \pm i\frac{d\chi}{dz} a_\pm \mp \omega b_\pm \,, \\ \frac{da_z}{dz} &= zI_1^0 + I_2^0, \quad \frac{dI_1^l}{dz} = \frac{i\omega}{v_z} I_1^l + S_l \,, \\ \frac{dI_2^l}{dz} &= \frac{i\omega}{v_z} I_2^l + Q_l - zS_l - \chi T_l \,, \\ \frac{dI_3^\pm}{dz} &= \frac{i\omega}{v_z} I_3^\pm + T_\pm \,. \end{aligned} \tag{16}$$

This system of equations with appropriate boundary conditions fully describes the interaction.

III. APPLICATIONS

A. The wiggler-free free electron laser

The problem is determined by specifying the form of the equilibrium magnetic field. An important case is when the equilibrium magnetic field is constant:

$$\frac{d\beta}{dz} = B_0 = \text{const.} \tag{17}$$

An analysis, similar to the present one, was carried out recently for this special case.[5] An instability occurs for waves at the frequency

$$\omega \cong \Omega/\gamma(1 - v_z) \,, \tag{18}$$

where Ω is the cyclotron frequency $eB_0/(mc)$. When the beam is randomly gyrophased, i.e.,

$$g_0 = 1, \quad g_n = 0, \quad n \neq 0 \,, \tag{19}$$

the instability is reduced to the cyclotron maser instability. However, a much higher growth rate is found when the beam is coherently gyrophased:

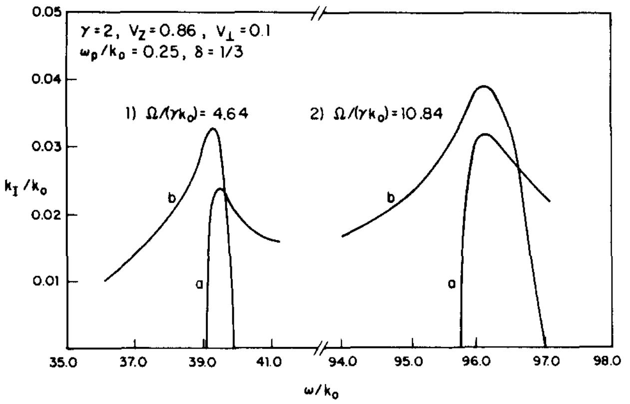

FIG. 1. Plot of normalized growth rate k_I/k_0 vs ω/k_0 for $\gamma = 2$, $v_z = 0.86$, $v_\perp = 0.1$, $\omega_p/k_0 = 0.25$, $\delta = 1/3$. (1) $\Omega/(\gamma k_0) = 4.64$; the first harmonic is mostly unstable. (2) $\Omega/(\gamma k_0) = 10.84$; the third harmonic is mostly unstable. (a) randomly gyrophased beam, (b) coherently gyrophased beam.

$$g_n = 1, \quad \forall n; \tag{20}$$

this corresponds to the wiggler-free free electron laser.

B. The lowbitron

We now apply this analysis to the lowbitron. The equilibrium magnetic field is

$$\frac{d\beta}{dz} = B_0[1 + \delta \sin(k_0 z)] . \tag{21}$$

The resonant frequencies are

$$\omega = (\Omega/\gamma + n k_0 v_z)/(1 - v_z) , \tag{22}$$

where n is a positive integer. The interaction was studied previously[1-3] for a randomly gyrophased beam of the form described by Eq. (19). The present analysis enables us to study the interaction of an arbitrarily gyrophased beam. We study in particular the coherently gyrophased beam of the form (20) and compare the growth rates of the instability for the two different types of beam. The problem we solve is the initial value problem. Given the values of the 13 unknowns at $z = 0$ we calculate their values for $z > 0$. We find that the growth rate of the instability is substantially larger when the coherently gyrophased beam is used than when the randomly gyrophased beam is used.

In order to demonstrate the gain enhancement caused by the use of the coherently gyrophased beam, we give two numerical examples. The growth rate of the instability is shown versus the frequency for the first harmonic and for the third harmonic. The parameters are the same as those in Ref. 3 and are chosen to maximize the growth rate for the first and third harmonics. The initial values of all the unknowns in Eq. (16) were set equal to zero except that of a_+. The system of equations (16) was solved numerically in the domain $0 < z \leqslant 200 \times 2\pi/k_0$. The growth rate k_I is defined as

$$k_I = \frac{d(\mathbf{a}^* \cdot \mathbf{a})/dz}{2(\mathbf{a}^* \cdot \mathbf{a})} . \tag{23}$$

We emphasize that even at the end of such a long interaction region several modes are coupled and the growth rate is not constant. Its value oscillates around some mean value, which is shown in Fig. 1. We see that in both cases the gain is enhanced by the use of the coherently gyrophased beam. For the first harmonic the maximum growth rate is increased by 36% and for the third harmonic by 21%.

The increased gain resulting from the use of a coherently gyrophased beam is due to the fact that usually the colder a beam is, the more unstable it is. A coherently gyrophased beam has zero temperature and is actually a cold fluid. The random gyrophase, similar to the thermal spread in other devices, reduces the gain.

In conclusion, the use of a coherently gyrophased beam is shown to substantially enhance the gain in the lowbitron. Since this beam is a crucial component in the wiggler-free free electron laser also, an effort to generate such a beam should be further encouraged.

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[1] W. A. McMullin and G. Bekefi, Appl. Phys. Lett. **39**, 845 (1981).
[2] W. A. McMullin and G. Bekefi, Phys. Rev. A **25**, 1826 (1982).
[3] R. C. Davidson and W. A. McMullin, Phys. Fluids **26**, 840 (1983).
[4] A. Fruchtman and L. Friedland, J. Appl. Phys. **53**, 4011 (1982).
[5] A. Fruchtman and L. Friedland, IEEE J. Quantum Electron. **QE19**, 327 (1983).
[6] P. Avivi, Ch. Cohen, J. L. Hirshfield, and A. Goldring, Bull. Am. Phys. Soc. **30**, 1539 (1985).