Electron heating by waves in the ion-cyclotron frequency range

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The interaction between waves in the ion-cyclotron frequency range and a hot-electron ring is studied. The configuration is approximated by a slab geometry in which the hot-electron ring is a few Larmor radii thick, the confining magnetic field is uniform, and the hot-electron distribution function is anisotropic. The wave vector of the incident wave is nearly perpendicular to the magnetic field with a small parallel component, so that both parallel streaming and finite Larmor radius effects are present. It is shown that a significant part of the energy flux may be absorbed by the electrons in a single pass of the wave through the ring. This process may be responsible for the electron ring heating observed in the Elmo Bumpy Torus-S [Nucl. Fusion 23, 49 (1983)] recently. Numerical results of energy absorption and wave transmission are presented for a wide range of parameters. For two special cases, the Born approximation and finite Larmor radius expansions enable us to simplify the analysis. The energy absorption calculated under the approximations is compared with the more exact treatment.

I. INTRODUCTION

In several experiments, externally produced waves on the order of the ion-cyclotron frequency propagate through hot-electron rings. In the presence of these waves, an increase of stored energy in the rings has recently been observed in the Elmo Bumpy Torus (EBT) experiment.[1] The purpose of this paper is to investigate one possible mechanism for this phenomenon of nonresonant heating of an anisotropic hot-electron ring by low-frequency waves. We examine the phenomenon by studying a simplified model problem. Viewing the ring as a hollow thin cylinder, we approximate the cylindrical surface as a plane, employ a rectangular geometry, and allow the equilibrium quantities to vary only along the x direction, which is perpendicular to this plane. The magnetic field is assumed to be uniform and to lie in the z direction. The equilibrium distribution function of the ring electrons is anisotropic and non-Maxwellian and is a function of the nonrelativistic constants of motion in the uniform magnetic field. The wave propagates mainly in the x direction and its wave vector has small y and z components. Both parallel streaming and finite Larmor radius effects are present in our configuration. We show that a considerable amount of the wave energy flux may be absorbed by the hot electrons during a single pass of the wave through the ring. This is one possible mechanism for the aforementioned observed heating of the electron rings. In the real device, precessional resonances and bounce-averaged motion may modify the mechanism we describe.

The interaction of the wave and the electrons is described within the framework of the Maxwell and Vlasov equations. We solve the scattering problem by constructing a solution composed of a given incoming wave, a reflected wave, and a transmitted wave. The rate of absorption is determined from the Poynting theorem.

The electron current is the first moment of the perturbed electron distribution function. The perturbed electron distribution function satisfies the linearized Vlasov equation, which is solved under the assumption that the wave frequency ω is much smaller than the electron-cyclotron frequency Ω. The resulting perturbed hot-electron current is mainly in the y direction, nearly perpendicular to both the magnetic field and the wave vector. Consequently, an integrodifferential equation for the electromagnetic X-mode wave is derived. We write an integral equation which is equivalent to the integrodifferential equation. The integral equation is solved by approximating it as a set of coupled algebraic equations.

Two regimes of parameters allow us to find approximate solutions. First, when the electron density is small, the Born approximation[2] is applied and an expression for the rate of absorption is derived. The Born approximation is based on the assumption that there is only a small change in the wave field because of its interaction with the hot electrons. Second, when the Larmor radius of the ring electrons is small relative to both the wavelength of the wave and the thickness of the slab, the expression for the current is simplified through a finite Larmor radius expansion. The integrodifferential equation becomes a second-order ordinary differential equation which is also solved numerically.

In Sec. II we solve the linearized Vlasov equation for the low-frequency regime and derive an expression for the current of the hot anisotropic electron ring. An integrodifferential equation for the wave electric field is obtained by substituting the expression for the current into Maxwell equations. Results of the Born approximation and the finite Larmor radius expansion are given in Sec. III and IV, respectively. Section V contains the solution of the general case. Results of the full solution are presented in Sec. VI and compared with the results obtained by the use of the two aforementioned approximations. We conclude in Sec. VII with suggestions for improving the model.

II. THE MODEL

In the model we employ, a constant equilibrium magnetic field $\mathbf{B} = B_0\hat{e}_z$ is present. The anisotropic equilibrium distribution function G is a function of the constants of mo-

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tion, the energy ϵ, and the canonical momentum p_y divided by the electron mass:

$$\epsilon = (\mathbf{v}\cdot\mathbf{v})/2, \quad p_y = v_y - x\Omega, \quad \Omega = eB_0/mc, \tag{1}$$

where $\mathbf{v}$ is the electron velocity, e and m are the electron charge and mass magnitudes, and c is the velocity of light in vacuum. Here G depends on x via p_y. The electric and magnetic wave fields have the form

$$\mathbf{E}' = \mathbf{E}(x)e^{i(k_y y + k_z z - \omega t)}, \quad \mathbf{B}' = \mathbf{B}(x)e^{i(k_y y + k_z z - \omega t)}. \tag{2}$$

The wave propagates mainly in the x direction. We Fourier-decompose the field with respect to the coordinates y and z only. The Vlasov equation, linearized about the given equilibrium, is

$$i(k_y v_y + k_z v_z - \omega)F + v_x \frac{\partial F}{\partial x} - \Omega\left(v_y \frac{\partial F}{\partial v_x} - v_x \frac{\partial F}{\partial v_y}\right)$$
$$= S \equiv (e/m)\{v_x[E_x G_{,\epsilon} - (B_z/c)G_{,p_y}] + v_y E_y G_{,\epsilon} + v_z[E_z G_{,\epsilon} + (B_x/c)G_{,p_y}] + E_y G_{,p_y}\}, \tag{3}$$

where the perturbed distribution function has the form

$$f_1(\mathbf{r},\mathbf{v},t) = F(x,\mathbf{v})e^{i(k_y y + k_z z - \omega t)}. \tag{4}$$

Here $G_{,p_y}$ and $G_{,\epsilon}$ are the partial derivatives of G with respect to p_y and ϵ. The characteristics of Eq. (3) are

$$v_x = v_x^0 \cos\Omega t - v_y^0 \sin\Omega t,$$
$$v_y = v_x^0 \sin\Omega t + v_y^0 \cos\Omega t,$$
$$x = v_y/\Omega - v_y^0/\Omega + x^0,$$

so that Eq. (3) is

$$i(\omega + k_y v_y + k_z v_z)F + \frac{\partial F}{\partial t} = S(t).$$

The unique periodic solution of (3) is

$$F = \frac{e^{-i[\Phi(t) + \Phi(t_0)]}}{e^{-i\Phi(0)} - e^{-i\Phi(t_0)}} \int_t^{t+t_0} dt'\, e^{i\Phi(t')}S(t'),$$

where

$$t_0 = 2\pi/\Omega,$$
$$\Phi(t) = (-\omega + k_z v_z)t - k_y v_x(t)/\Omega.$$

More explicitly, we have

$$F(t) = \{\exp[2\pi i(-\omega + k_z v_z)/\Omega] - 1\}^{-1} \times \int_0^{t_0} dt'\, S(t+t')\exp\{i(-\omega + k_z v_z)t' - (k_y/\Omega)[v_x(t+t') - v_x(t)]\}, \tag{5}$$

where S, the inhomogeneous part of Eq. (3), is calculated along the characteristics

$$v_x(t+t') = v_x(t)\cos\Omega t' - v_y(t)\sin\Omega t',$$
$$v_y(t+t') = v_x(t)\sin\Omega t' + v_y(t)\cos\Omega t', \tag{6}$$
$$x(t+t') = (1/\Omega)[v_y(t+t') - p_y(t)].$$

Since we are dealing with the interaction of electrons with ion-cyclotron frequency waves, ω is much smaller than Ω. Furthermore, the wave vector of the incident wave is nearly parallel to the x direction so that $k_y v_y$ and $k_z v_z$ are also much smaller than Ω. Thus, one can approximate (5) and write

$$F = \frac{\Omega}{2\pi i(-\omega + k_z v_z)} \int_0^{t_0} dt'\, S(t+t'). \tag{7}$$

Finite Larmor radius effects which have been neglected in the y and z directions are not neglected in the x direction, the approximate direction of the wave propagation. In addition, the anisotropy in the distribution function may introduce a strong x dependence to the wave. Nevertheless, in the EBT-S experiment the wave may have a relatively high poloidal $\mathbf{k}$ component (k_y in our model). The treatment could be modified to deal with such a case. Let us consider one term that appears in (7) which comes from the terms proportional to v_x in the definition of S [see (3)],

$$I = \int_0^{t_0} dt'\, v_x(t+t')\{E_x[x(t+t')]G_{,\epsilon} - (1/c)B_z[x(t+t')]G_{,p_y}\}. \tag{8}$$

Since $x(t+t')$ is a function of $v_y(t+t')$ only and

$$v_x(t+t') = v_\perp \cos\Omega t'', \quad v_y(t+t') = v_\perp \sin\Omega t'', \tag{9}$$

we see that I is proportional to $\int_0^{t_0} dt''\cos\Omega t'' f(\sin\Omega t'')$, which is zero. Thus, we may drop these terms in (3) and (7).

For simplicity, we assume that E_z, the component of the electric field parallel to $\mathbf{B}_0$, is zero. The term σ_{zz} in the cold plasma conductivity tensor associated with the background cold electrons is so large that E_z must be very small.[3] Provided the density of the hot electrons is small relative to that of the cold, we may neglect E_z, restricting ourselves to the extraordinary mode. Thus (7) reduces to the following expression for the distribution function:

$$F = \frac{\Omega}{2\pi i(-\omega + k_z v_z)} \int_0^{t_0} dt' \frac{e}{m}\{v_y E_y G_{,\epsilon} + [v_z(B_x/c) + E_y]G_{,p_y}\}. \tag{10}$$

The Maxwell equations and the assumption that E_z is zero yield

$$B_x = -(k_z c/\omega)E_y, \tag{11}$$

so that F becomes

$$F = \frac{\Omega}{2\pi i(-\omega + k_z v_z)} \frac{e}{m} G_{,\epsilon} \int_0^{t_0} dt'\, v_y(t+t')E_y(t+t') - \frac{\Omega}{2\pi i\omega} \frac{e}{m} G_{,p_y} \int_0^{t_0} dt'\, E_y(t+t'). \tag{12}$$

We may calculate the perturbed currents as first moments of F given by (12). Since $v_y(t+t')$ and $x(t+t')$ are functions of $v_y(t)$ and $(v_x^2 + v_y^2)(t)$, J_x vanishes after the v_x integration.

We now specify the distribution function. For simplicity and to permit relatively explicit analysis, we choose the specific equilibrium distribution function

$$G = K\exp(-\lambda\epsilon - \mu p_y^2), \tag{13}$$

from which the density follows by integrating over $\mathbf{v}$:

$$n(x) = n_0 \exp\left(-\frac{\mu\lambda\Omega^2}{2(\lambda + \mu)}x^2\right), \quad n_0 = K\left(\frac{(2\pi)^3}{\lambda^2(\lambda + \mu)}\right)^{1/2}. \tag{14}$$

We may then introduce the thermal velocity $v_{\rm th}$, Larmor radius r, and effective ring thickness d as

$$v_{\rm th} = (2/\lambda)^{1/2}, \quad r = v_{\rm th}/\Omega, \quad d = [2(\lambda+\mu)/\mu\lambda\Omega^2]^{1/2}. \tag{15}$$

The current is defined as

$$J_y = -e\int d\mathbf{v}\, v_y F. \tag{16}$$

We start with the v_z integration. Integrating the first term of F, we obtain

$$\int_{-\infty}^{\infty} dv_z \frac{\exp(-v_z^2/v_{\rm th}^2)}{(-\omega + k_z v_z)}$$
$$= \text{P.V.}\left[\int_{-\infty}^{\infty} \frac{dv_z}{|k_z|(v_z - v_B)} \exp\left(\frac{-v_z^2}{v_{\rm th}^2}\right)\right]$$
$$+ (i\pi/|k_z|)\exp(-v_B^2/v_{\rm th}^2) = \pi^{1/2} Z(v_B/v_{\rm th})/|k_z|.$$

Here Z is the plasma dispersion function and v_B is ω/k_z. Use has been made of the fact that Im $\omega > 0$. The current J_y becomes

$$J_y = -\frac{e^2\Omega\sqrt{\pi}}{2m\pi i|k_z|} Z\left(\frac{v_B}{v_{\rm th}}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dv_x\, dv_y\, v_y\, g_{,\epsilon}$$
$$\times \int_0^{t_0} dt'\, v_y(t+t')E_y[x(t+t')]$$
$$+ \frac{e^2\Omega}{2m\pi i\omega}\sqrt{\pi}\, v_{\rm th}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dv_x\, dv_y\, v_y\, g_{,p_y}$$
$$\times \int_0^{t_0} dt'\, E_y[x(t+t')], \tag{17}$$

where $G = g\exp(-v_z^2/v_{\rm th}^2)$.

In order to simplify the expression for J_y we apply successive changes of variables. First we use cylindrical coordinates in velocity space. In the (v_x, v_y) plane, $v_y(t)$ is $v\cos\tilde{\theta}$ and $v_y(t+t')$ is $v\cos\tilde{\phi}$. The integration of t' becomes an integration on $\tilde{\phi}$. At this stage $x(t+t')$ is

$$x(t+t') = x(t) - \frac{2v}{\Omega}\sin\left(\frac{\tilde{\phi}+\tilde{\theta}}{2}\right)\sin\left(\frac{\tilde{\phi}-\tilde{\theta}}{2}\right). \tag{18}$$

The next change of variables is as follows:

$$\phi \equiv \tilde{\phi} - \tilde{\theta}, \quad \theta \equiv (\tilde{\theta} + \tilde{\phi})/2, \tag{19}$$

$$\tilde{v}_x \equiv v\sin\theta, \quad \tilde{v}_y \equiv v\cos\theta. \tag{20}$$

In these new variables,

$$x(t+t') = x - [2\tilde{v}_x \sin(\phi/2)]/\Omega, \tag{21}$$

which does not depend on $\tilde{v}_y$. The substitution

$$x' \equiv (2\tilde{v}_x \sin\alpha)/\Omega, \quad \alpha = \phi/2, \tag{22}$$

leaves x' as the only variable of integration on which the unknown quantity E_y depends. Upon performing the integration, we find that the current is

$$J_y(x) = -i\frac{\omega_p^2}{8\pi^3}\left(\frac{\Omega}{\omega}\right)\frac{\pi^{1/2}}{v_{\rm th}}(1+s)^{1/2}\exp\left(\frac{-x^2}{d^2}\right)$$
$$\times \int_0^{\pi} \frac{d\alpha}{\sin\alpha(1+s\cos^2\alpha)^{3/2}}$$
$$\times \int_{-\infty}^{\infty} dx'\exp(-\beta^2)E_y(x-x')$$
$$\times\left[\frac{v_B}{v_{\rm th}} Z\left(\frac{v_B}{v_{\rm th}}\right)\left(\cos^2\alpha + \frac{2s^2}{r^2}\frac{(x'/2-x)^2\cos^4\alpha}{(1+s\cos^2\alpha)}\right.\right.$$
$$\left. - \frac{x'^2}{2r^2}(1+s\cos^2\alpha)\right)$$
$$\left. - s\left(\cos^2\alpha + \frac{2}{r^2}\frac{(x'/2-x)(x'/2+sx\cos^2\alpha)}{(1+s\cos^2\alpha)}\right)\right]. \tag{23}$$

Here $s \equiv \mu/\lambda = r^2/(d^2 - r^2)$, $\omega_p^2 = 4\pi n_0 e^2/m$, $\beta = (x' - x\zeta)/\delta$, $\delta = 2r\sin\alpha[(1+s\cos^2\alpha)/(1+s)]^{1/2}$, and $\zeta = 2s\sin^2\alpha/(1+s)$. We then proceed to substitute this current into Maxwell's equations.

Maxwell's equations in our case become

$$k_z^2 E_x = (\omega/c)^2(\epsilon_1 E_x + i\epsilon_2 E_y), \tag{24}$$
$$k_z^2 E_y - \frac{d^2 E_y}{dx^2} = \left(\frac{\omega}{c}\right)^2(-i\epsilon_2 E_x + \epsilon_1 E_y) + \frac{4\pi i\omega}{c^2}J_y,$$

where we have used the assumptions $E_z = 0$, $k_y = 0$, and $J_x = 0$. Here, J_y is the current from the hot electrons given in (23), and ϵ_1 and ϵ_2 are the diagonal elements of the cold plasma dielectric tensor.[4] The first of the equations (24) enables us to write E_x as a function of E_y. We assume that Ω and ω_{pe} (the electron plasma frequency) are of the same order of magnitude. Since ω is much smaller than both Ω and ω_{pi} (the ion plasma frequency),

$$\epsilon_1 \cong \omega_{pi}^2/(\Omega_i^2 - \omega^2), \tag{25}$$
$$\epsilon_2 \cong -(\omega/\Omega_i)[\omega_{pi}^2/(\Omega_i^2 - \omega^2)],$$

where Ω_i is the ion-cyclotron frequency. Thus Eqs. (24) and (25) become

$$\frac{d^2E_y}{dx^2} + k^2 E_y = -\frac{4\pi i\omega}{c^2}J_y \equiv D(x), \tag{26}$$
$$k^2 \equiv \frac{\omega^2}{\Omega_i^2}\frac{\omega_{pi}^2}{c^2} - k_z^2 \cong \frac{\omega^2}{\Omega_i^2}\frac{\omega_{pi}^2}{c^2}.$$

Before we continue with the solution of this general case, we consider separately two approximations: the Born approximation and a finite Larmor radius expansion.

III. THE BORN APPROXIMATION

The Born approximation[2] relies upon the assumption that there is only a small change in the radiation fields because of their interaction with the electron ring. This allows one to use perturbation methods in order to find the fields. To lowest order, the wave fields in the expressions for the currents are assumed to be the given incident wave fields. One can substitute the calculated currents into the Maxwell equations and solve for the higher-order fields. Being interested mainly in the calculation of the heating, we use a slightly different procedure. We do calculate the currents by assuming that the wave fields are the incident wave fields. The rate of heating, which is $\mathrm{Re}(\mathbf{E}^*\cdot\mathbf{J})$, is found by using the incident wave field for $\mathbf{E}$ and the approximate current for $\mathbf{J}$. We do not solve the Maxwell equations for the higher-order fields, since this is not necessary for the lowest-order heating calculation.

The incident wave is of the form

$$\mathbf{E} = \hat{e}_y e^{ikx}. \tag{27}$$

For our convenience, we use Eq. (17) for the current. Substituting expression (27) for $\mathbf{E}$, we obtain the rate of absorption

$$\mathrm{Re}(E_y^* J_y) = -\frac{e^2\Omega\sqrt{\pi}}{2m\pi|k_z|}\mathrm{Im}\left[Z\left(\frac{v_B}{v_{\mathrm{th}}}\right)\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dv_x\, dv_y\, v_y\, g_{,\epsilon} \times e^{-ikv_y/\Omega}\int_0^{t_0} dt'\, v\cos\Omega t' e^{+i(kv/\Omega)\cos\Omega t'}, \tag{28}$$

where one term vanished because of the t' integration. We change now to cylindrical coordinates, so that $dv_x\, dv_y = v\, dv\, d\phi$ and $v_y = v\cos\phi$. Using the identities[5]

$$\int_0^{2\pi} e^{iz\cos\theta}\cos\theta\, d\theta = i\frac{\pi}{2}J_1(z), \quad J_1(-z) = -J_1(z),$$

and the explicit form of the Z function, we find that the total absorption is

$$P = \int_{-\infty}^{\infty} dx\, \mathrm{Re}(E_y^* J_y) = \frac{\omega_p^2 d}{\omega}\rho\exp(-\rho^2)\int_0^{\infty} d\zeta\, \zeta^3 e^{-\zeta^2} J_1^2(kr\zeta) = (\omega_p^2\, d/\omega)\rho\exp(-\rho^2)(kr)^2/4\; {}_1F_1\left[\tfrac{3}{2},2,-(kr)^2\right]. \tag{29}$$

Here J_1 and ${}_1F_1$ are the Bessel function and the confluent hypergeometric function,[6] respectively, and $\rho \equiv v_B/v_{\mathrm{th}}$. We divide P by the energy flux expressed by the magnitude of the Poynting vector

$$N = (kc^2/4\pi\omega)|Ey|^2 \tag{30}$$

to obtain the relative absorption

$$P_R = (\omega_p^2/c^2)\pi\, dk\, r^2\rho\exp(-\rho^2)\,{}_1F_1\left[\tfrac{3}{2},2,-(kr)^2\right]. \tag{31}$$

If we define the total mass as $n_t = \int_{-\infty}^{\infty} dx\, n(x) = n_0\pi^{1/2}d$, expression (31) shows that the absorption does not depend on the slab thickness d, but on the total mass only.

The absorption originates from both parallel streaming and finite Larmor radius effects. The maximum absorption occurs when $\rho = 1/\sqrt{2}$, that is, when ω/k_z is close to the thermal velocity of the electrons. When ρ is much larger or smaller than one the absorption disappears. Clearly the absorption results from the interaction of the wave having a component of propagation parallel to the magnetic field with the hot electrons that have a velocity equal to v_B. This absorption, which is a Landau-damping type, is highest when the velocity of the resonant electrons is the thermal velocity. The anisotropy of the ring does not seem to affect the absorption, at least not in the Born approximation.

When $kr \ll 1$ one can approximate ${}_1F_1$ as

$${}_1F_1\left[\tfrac{3}{2},2,-(kr)^2\right] = 1. \tag{32}$$

Thus

$$P_R = (\omega_p^2/c^2)\pi\, dk\, r^2\rho\exp(-\rho^2). \tag{33}$$

IV. FINITE LARMOR RADIUS (FLR) EXPANSIONS

The assumption upon which this expansion relies is that the Larmor radius r is much smaller than other lengths in the system, including $(2\pi/k)$ and d. One can expand the integrands in expression (23) for the current and perform the integrations. The zeroth-order and the first-order terms of the expansion in r vanish on the integration on α or β. The second-order term is the lowest-order term which has a finite contribution. The current in this approximation is

$$J_y(x) = \frac{i\omega_p^2}{4\pi}\frac{r^2}{\omega}\exp\left(-\frac{x^2}{d^2}\right)\left[\rho Z(\rho)\left(\frac{d^2E_y}{dx^2} + \frac{x}{d^2}\frac{dE_y}{dx}\right) + \frac{E_y}{d^2}\left(1-\frac{2x^2}{d^2}\right) + \frac{x}{d^2}\frac{dE_y}{dx}\right]. \tag{34}$$

Substituting this expression for the current in Eq. (26), one obtains a second-order ordinary differential equation for E_y:

$$\frac{d^2E_y}{dx^2} + k^2E_y = \frac{\omega_p^2}{c^2}r^2\exp\left(-\frac{x^2}{d^2}\right)\left[\rho Z(\rho)\left(\frac{d^2E_y}{dx^2} + \frac{x}{d^2}\frac{dE_y}{dx}\right) + \frac{E_y}{d^2}\left(1-\frac{2x^2}{d^2}\right) + \frac{x}{d^2}\frac{dE_y}{dx}\right]. \tag{35}$$

This equation can be solved with an appropriate set of boundary conditions. The physical problem we wish to solve is the scattering problem, for which the asymptotic boundary conditions are

$$E_y(x) = e^{ikx} + Re^{-ikx}, \quad x\to-\infty, \qquad E_y(x) = Te^{ikx}, \quad x\to\infty. \tag{36}$$

Defining $E(x) = (1/T)E_y(x)e^{-ikx}$, we obtain from Eq. (35) a second-order ordinary differential equation for $E(x)$ with the boundary conditions

$$E(x) = 1, \quad \frac{dE(x)}{dx} = 0, \quad x\to\infty. \tag{37}$$

This equation can be solved numerically. The asymptotic solution is

$$E(x) = 1/T + (R/T)e^{-2ikx}, \quad x\to-\infty, \tag{38}$$

which is an oscillatory solution of average amplitude $1/|T|$ and an oscillation amplitude $|R|/|T|$. The constants T and R may be found once the solution (38) has been plotted. Applying the Poynting theorem, we have for the rate of absorption

$$P_R = 1 - |R|^2 - |T|^2. \tag{39}$$

In Sec. VI, numerical solutions of Eq. (35) are presented and compared with the solutions of the general equation.

V. THE GENERAL SOLUTION

Equation (26) together with expression (23) for J_y is an integrodifferential equation for E_y. The general solution of this Helmholtz equation is

$$E_y = Ae^{ikx} + Be^{-ikx} + \int_{-\infty}^{\infty} dx'\, G(x,x')D(x'), \qquad (40)$$

where $G(x,x')$ is the one-dimensional Green function

$$G(x,x') = e^{ik|x-x'|}/(2ik). \qquad (41)$$

Since we are looking for a solution for the scattering problem of the form (36), where there is no incident wave coming from infinity, we choose $A = 1$ and $B = 0$ in Eq. (40).

The source function D depends on E_y, that is, Eq. (40) is an integral equation. This is an inhomogeneous Fredholm equation of the second type. It can be rewritten in the form

$$E_y(x) = e^{ikx} + \int_{-\infty}^{\infty} dx'\, K(x,x')E_y(x'), \qquad (42)$$

where the kernel is

$$\begin{aligned} K(x,x') = &-\left(\frac{\omega_p^2}{c^2}\right)\frac{(1+s)^{1/2}}{\pi^{3/2}r}\int_{-\infty}^{\infty} d\xi\, G(x,\xi)\\ &\times\int_0^1 \frac{dy}{(1-y^2)(1+sy^2)^{3/2}}\\ &\times\exp\left[-\frac{1}{4r^2}\left(\frac{(\xi-x')^2}{(1-y^2)} + \frac{s(\xi+x')^2}{(1+sy^2)}\right)\right]\\ &\times\Bigg[\rho Z(\rho)\left(y^2 + \frac{s^2}{2r^2}\frac{(\xi+x')^2y^4}{(1+sy^2)}\right.\\ &\left.-\frac{(\xi-x')^2}{2r^2}(1+sy^2)\right)\\ &-s\left(y^2 + \frac{(\xi+x')(\xi-x'-2sx'y^2)}{2r^2(1+sy^2)}\right)\Bigg]. \qquad (43)\end{aligned}$$

Each term in this kernel is a double integral. The integrand is singular at $y = 1$ and behaves as $(1-y)^{-1/2}$. A standard routine which overcomes such singularities at the end points was employed. Since the derivative of the Green function is discontinuous, the infinite regime was divided into two parts. The integral on the semi-infinite regimes was found by applying the Gauss–Laguerre method. When the parameters were such that an integral over an infinite regime was required, the Gauss–Hermite method was used.

Equation (42) is solved by approximating it as a set of coupled linear algebraic equations. The part of the x axis, where the kernel is not too small, is split into N (= 40) intervals of equal lengths. Denoting their limits as x_i and substituting $E = E_y e^{ikx}$, Eq. (42) becomes

$$E(x_j) = 1 + \Delta x \sum_{i=1}^{N} \exp[ik(x_j - x_i)]K(x_j,x_i)E(x_i),$$
$$j = 1,\ldots,N, \quad \Delta x = x_{i+1} - x_i. \qquad (44)$$

For large values of j, $|E(x_j)|^2 = |T|^2$, and for small values of j, $|E(x_j) - 1|^2 = |R|^2$. Again, by applying the Poynting theorem, P_R can be found [Eq. (39)].

In the next section numerical solutions of Eq. (44) are presented and compared with the results of the aforementioned approximation methods.

VI. NUMERICAL EXAMPLES

We present three examples. The first shows the absorption in a parameter regime where the Born approximation is valid. The second demonstrates the validity of the small Larmor radius expansion for a different regime of parameters. In the last example we study a case where both approximations fail and the correct amount of absorption is found only by solving the general equation.

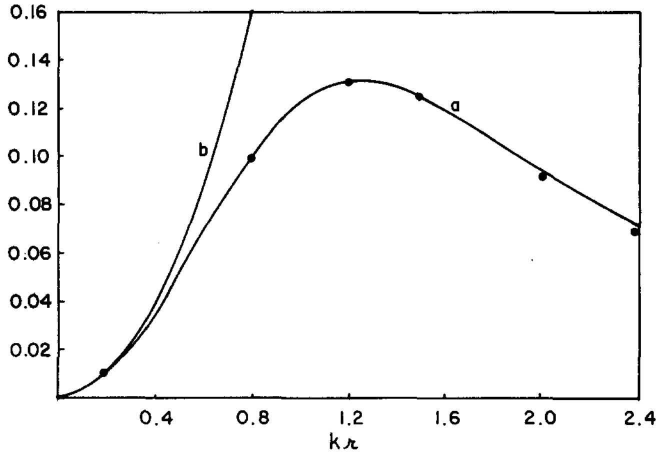

FIG. 1. The function $(kr/2)^2\,{}_1F_1[\frac{3}{2}, 2, -(kr)^2]$ vs kr [curve (a)]; $r = 0.3$ cm, $d = 10r$. The parabola [curve (b)] is $(kr/2)^2$ and the dots are the results of the exact equation. The electron density was changed in order to keep the absorption rate small, while solving the exact equation. In the regime of small absorption but high values of kr ($\gtrsim 0.4$), the FLR expansion fails and the Born approximation is valid.

In all the cases we treat, the reflection coefficient is much smaller than the transmission coefficient so that the rate of absorption is practically a function of the transmission coefficient. We do not give the small reflection coefficient which is on the order of one percent.

In Fig. 1 the function $(kr/2)^2\,{}_1F_1[\frac{3}{2}, 2, -(kr)^2]$ is plotted versus kr [curve (a)]. This function, multiplied by $(\omega_p^2/c^2)\pi\, d\,(4/k)\rho\exp(-\rho^2)$, gives the relative absorption, according to the Born approximation [see Eq. (31)]. The dots in this figure express the relative absorption calculated by the results of the exact equation (44) and divided by the factor $(\omega_p^2/c^2)\pi\, d\,(4/k)\rho\exp(-\rho^2)$. The Larmor radius r is 0.3 cm and the slab thickness d is ten times larger. In solving this equation ω_p^2 was given different values for different values of

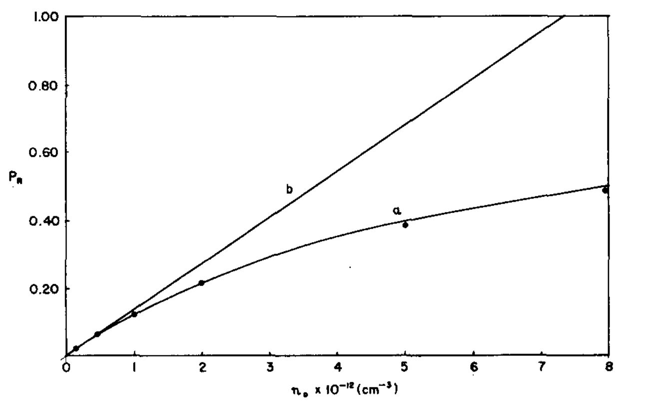

FIG. 2. The relative absorption P_R versus the electron density n_0; $r = 0.15$ cm, $d = 20r$, $2\pi/k = 100r$. Curves (a) and (b) show the results of the FLR expansion and the Born approximation, respectively, while the dots are the results obtained by solving the exact equation. Here the FLR expansion is valid, while the Born approximation fails for densities higher than 1×10^{12} cm^{-3}.

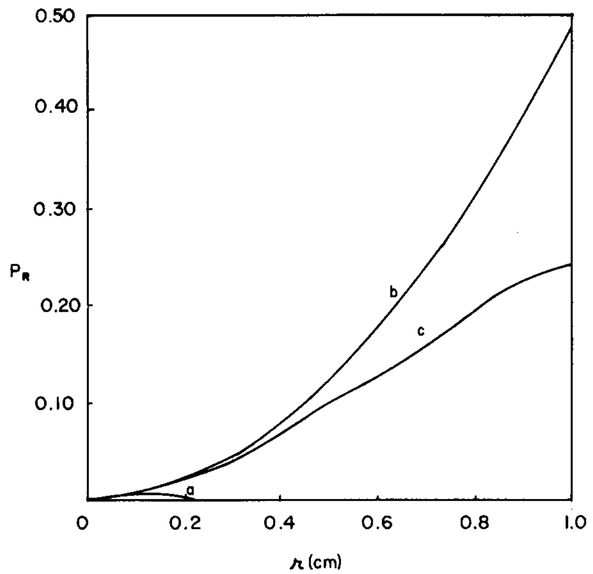

FIG. 3. The relative absorption P_R versus the Larmor radius r. The parameters are $d = 1$ cm, $n_0 = 5\times10^{11}$ cm^{-3}, $2\pi/k = 30$ cm. (a) FLR expansion, (b) the Born approximation, and (c) the solution of the exact equation. For large r both approximations fail.

kr in a way that the absorption was kept less than 5%. As can be seen in the figure, the relative absorptions calculated by the Born approximation and by the solution of the exact equation are the same. This shows the validity of the Born approximation in the case where the absorption is small (less than 5% in our example). As expected, the values of kr do not affect the validity of the Born approximation as long as the absorption is small. Curve (b) in Fig. 1 shows results obtained by the finite Larmor radius expansion method. We see that for $kr \lesssim 0.4$ the FLR expansion yields correct results but for higher kr values this method fails.

In Fig. 2, d is 20 times larger than r and λ $(=2\pi/k)$ is 100 times larger than r. This parameter regime is suitable for the application of the finite Larmor radius expansion method. Curve (a) shows the relative absorption found through the FLR expansion [Eq. (35)] versus the density n_0 of the hot electrons. The dots are the solutions of the exact equation [Eq. (44)]. The agreement between the solutions of the two equations proves the validity of the finite Larmor radius expansion method in this case. When the Larmor radius is small relative to other lengths in the system, the FLR expansion method is valid for all densities. Curve (b) shows the absorption calculated by the Born approximation [expression (31)]. For densities higher than 1×10^{12} cm^{-3} the Born approximation fails in this case. The Born approximation is correct only when the relative absorption is small enough.

Figure 3 shows the relative absorption versus the Larmor radius r. Curve (a) shows the results of the finite Larmor radius expansion [Eq. (35)], curve (b) those of the Born approximation [Eq. (31)], and curve (c) the results of solving the exact equation (44). The slab thickness d is 1 cm, λ is 30 cm, and the electron density n_0 is 5×10^{11} cm^{-3}. This is an example for a parameter regime where none of the approximations is correct. When r is larger than 0.1 cm, it is not small enough relative to d, and the FLR expansion fails. When r is about 0.4 cm, the relative absorption is too large (nearly 10%), and the Born approximation fails. Then only the solution of the exact equation gives the correct amount of absorption.

The experimental parameters in the EBT-S experiment[1] probably fit roughly the case of $r = 0.5$ cm in Fig. 3.

VII. CONCLUSIONS

The results presented here suggest that nonresonant heating of electrons by a low-frequency radiation may be substantial. We chose an approximate model in order to simplify the analysis. Future studies may analyze a more realistic model in order to get more accurate results.

The form of the magnetic field can include spatial gradients which are present in fusion devices. The finite poloidal dimension introduces finite k_y component to the wave vector. Furthermore the slab model may be modified to represent a more realistic geometry. The wave electric field component parallel to the magnetic field may be comparable to the perpendicular component when the fraction of the hot electrons is not small. These nonvanishing k_y and E_z couple the various components of the wave fields and may even enhance the heating. Thus a relativistic model of the hot electrons can be used to explore additional effects that the present nonrelativistic model could not deal with.

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