Theory of a Nonwiggler Collective Free Electron Laser in Uniform Magnetic Field

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***Abstract*—A nonwiggler free electron laser, operating in uniform guide magnetic field, is analyzed. The amplifier problem is solved self-consistently on the basis of the kinetic theory. It is shown that the asymmetry in the azimuthal distribution of the electrons' momentum leads to a coupling between the transverse and the space-charge modes. This, in turn, enhances the gain in the amplifier. In the case of a cold beam, with the electrons gyrating coherently, the spatial growth in the collective nonwiggler free electron laser (FEL) is comparable to that found in conventional free electron lasers operating in similar regimes.**

I. Introduction

Conventional free electron lasers (FEL's) explore the idea of backscattering of a low-frequency pump wave by relativistic electron beams. The pump wave forces the beam to oscillate coherently, resulting in possible stimulated emission at a wavelength shorter by roughly a factor $\alpha\gamma^2$ (γ being the relativistic factor of the beam) than the wavelength λ_0 of the pump wave [1]. For $\gamma^2 >> 1$ the coefficient α is 4 or 2, depending on whether the pump is a regular electromagnetic wave or a magnetostatic spatially periodic field. The latter is typically produced on the axis of a magnetic wiggler (a bifilar helical current winding with equal and opposite currents in each helix).

Since the first successful operation of FEL at Stanford University [2], wigglers became an integral part in most FEL experiments. Nonetheless, both theory [3], [4] and experiments [5] showed that special care should be taken in constructing wigglers and in choosing radial dimensions and entrance conditions of the beam in order to observe coherent helical electron orbits in the laser. Together with this it was appreciated recently that spatially coherent undulation of the beam, and therefore also Doppler upshifted stimulated emission, can be caused not only by a wiggler but also by the natural gyration of the electron beam in a uniform guide magnetic field. In fact, in a cold beam, the electrons move on coherent helical orbits with the pitch $\lambda_0 = 2\pi\gamma u/\Omega$

Manuscript received June 11, 1982; revised October 12, 1982. This work was supported in part by the U.S. Office of Naval Research and by the U.S. Israel Binational Science Foundation.

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0018-9197/83/0300-0327$01.00

where Ω is the nonrelativistic cyclotron frequency, characterizing the guide field, and u is the velocity of the electrons in the direction of the guide field. Thus, amplification in the system is expected at a wavelength $\lambda = \lambda_0/2\gamma^2 = \pi u/\gamma\Omega$. Single particle calculations of small and large signal gains in such a nonwiggler FEL [6], [7] confirmed the attractive possibility of replacing the wigglers by a uniform guide field, which was, as a matter of fact, almost always present in conventional FEL experiments. However, the encouraging predictions of the single particle theories could not be applied to the collective regime of operation, when intense electron beams ($I > 1$ kA) with relatively low energies ($\gamma < 10$) were used. The collective interaction, usually termed "stimulated Raman scattering," had to be treated by a self-consistent theory.

Ott and Manheimer published a theory for a thin slab beam in a waveguide [8]. The first study of a nonwiggler FEL operating in the collective regime in free space was given by Hirshfield *et al.* [9], who showed the existence of a spatially unstable mode at the Doppler upshifted cyclotron frequency. They considered a randomly gyrophased electron beam, in which the momentum distribution function of the electrons was

$$f(p_\perp, p_z, \phi) = f(p_\perp, p_z). \tag{1}$$

Here $p_\perp$ and p_z are the momentum components, perpendicular and parallel to the direction of the guide field, and ϕ is the azimuthal angle ($t_g\,\phi = p_y/p_x$). It was also shown in [9] that the longitudinal and transverse modes of the system are decoupled and only the azimuthal bunching mechanism drives a cyclotron maser type instability. In this respect, the device, considered in [9] and based on randomly gyrophased beams, differs significantly from the conventional FEL's, where the axial density bunching is primarily responsible for the spatial instability.

We now show that the mode decoupling described in [9] is the result of the random gyrophase distribution of the electrons in the beam. We write the distribution function in the form

$$f(\boldsymbol{p}, z, t) = \tilde{f}(\boldsymbol{p}, z, t)\, \tilde{N}(z, t) \tag{2}$$

where

$$\int \tilde{f}(\boldsymbol{p}, z, t)\, d^3\boldsymbol{p} = 1. \tag{3}$$

Let $\tilde{f} = \tilde{f}_0 + \tilde{f}_1$ and $\tilde{N} = \tilde{N}_0 + \tilde{N}_1$ where $\tilde{f}_0$ and $\tilde{N}_0$ are the values of $\tilde{f}$ and $\tilde{N}$ when there are no perturbing electromagnetic fields. Then the linearized perturbed transverse electron current, which is the source in the Maxwell equations for the transverse fields, is written as

$$\boldsymbol{J}_\perp = e\tilde{N}_0 \langle \boldsymbol{v}_{\perp 1} \rangle + e \langle \boldsymbol{v}_{\perp 0} \rangle \tilde{N}_1 \tag{4}$$

where

$$\langle \boldsymbol{v}_{\perp 1} \rangle = \int \boldsymbol{v}_\perp \tilde{f}_1\, d^3\boldsymbol{p}, \quad \langle \boldsymbol{v}_{\perp 0} \rangle = \int \boldsymbol{v}_\perp \tilde{f}_0\, d^3\boldsymbol{p} \tag{5}$$

and the subscript $\perp$ denotes components transverse to the guide field. In the case of the random gyrophase distribution (1), $\langle \boldsymbol{v}_{\perp 0} \rangle$ vanishes and, as a result, only the transverse velocity perturbation $\langle \boldsymbol{v}_{\perp 1} \rangle$ contributes to $\boldsymbol{J}_\perp$. If, however, the momentum distribution has an azimuthal asymmetry, then $\langle \boldsymbol{v}_{\perp 0} \rangle \neq 0$ and the density modulation $\tilde{N}_1$ (or the axial density bunching) can also drive transverse modes of the system.

An example of a nonwiggler FEL with an azimuthally asymmetric electron beam was recently studied in [10]. The beam was assumed to be cold, and the momentum distribution at the entrance into the device was taken to be

$$f(p_\perp, p_z, \phi) = \frac{N_0}{2\pi p_\perp}\, \delta(p_\perp - p_{\perp 0})\, \delta(p_z - p_{z0})\, \delta(\phi - \phi_0). \tag{6}$$

We will use the term "helical beam" to describe such a beam configuration. It was demonstrated in [10] that in a laser the helical beam provides enhanced spatial gain compared to that found with a randomly gyrophased electron beam. The origin of the gain enhancement is the aforementioned increased role of the axial bunching in driving the instability.

This paper presents a kinetic theory of nonwiggler FEL's in a uniform guide magnetic field. We consider an arbitrary ϕ dependence of the electron momentum distribution function and, in contrast to the cold fluid model of [10], we base the theory on a Maxwell-Vlasov description. The scope of the paper is as follows. In Section II the Maxwell equations are reduced to a simple set of first order ordinary differential equations for the electric component of the electromagnetic field in the system. The current and density sources for the field equations are found in Section III. In Section IV we apply the Laplace transformation to the field equations and derive the dispersion relation governing the stability of our system. Finally, in Section V the dispersion relation is studied numerically for several configurations of the electron beam. In the same section we also solve the field equations directly and find the actual gain in a finite length nonwiggler FEL amplifier.

II. Field Equations

Consider an electromagnetic wave propagating along a relativistic electron beam, gyrating in a uniform magnetic field $\mathcal{B} = \mathcal{B}_0 \hat{e}_z$. Assuming a one-dimensional model, we can describe the electromagnetic fields $\boldsymbol{E}(z, t)$ and $\boldsymbol{B}(z, t)$ by the system of Maxwell equations

$$c\hat{e}_z \times \frac{\partial \boldsymbol{B}_\perp}{\partial z} = \frac{\partial \boldsymbol{E}_\perp}{\partial t} + 4\pi \boldsymbol{J}_\perp \tag{7}$$

$$-c\hat{e}_z \times \frac{\partial \boldsymbol{E}_\perp}{\partial z} = \frac{\partial \boldsymbol{B}_\perp}{\partial t} \tag{8}$$

$$\frac{\partial E_z}{\partial z} = -4\pi e N \tag{9}$$

$$B_z = 0 \tag{10}$$

where $\boldsymbol{J}_\perp$ and N are the self-consistent transverse current and electron density perturbations caused by the presence of the electromagnetic wave.

We restrict our analysis to the stationary amplifier problem, namely, we introduce an electromagnetic perturbation of frequency ω at $z = 0$ and solve for the electromagnetic fields at given $z > 0$. Respectively, we write

$$\boldsymbol{E}(z, t) = \mathrm{Re}\left[\frac{mc^2}{e}\, \boldsymbol{a}(z)\, \Phi\right] \tag{11}$$

$$\boldsymbol{B}(z,t) = \mathrm{Re}\left[\frac{mc^2}{e}\,\boldsymbol{b}(z)\,\Phi\right] \tag{12}$$

$$\boldsymbol{J}_\perp(z,t) = \mathrm{Re}\left[\frac{m}{4\pi e^2}\,\boldsymbol{j}_\perp(z)\,\Phi\right] \tag{13}$$

$$N(z,t) = \mathrm{Re}\left[\frac{m}{4\pi e^2}\,n(z)\,\Phi\right] \tag{14}$$

where

$$\Phi = \exp\left[i\,\frac{\omega}{c}\,(z-ct)\right]. \tag{15}$$

Note that consistent with the amplifier problem we left in (11)-(14) only waves propagating in the positive z-direction, which is also the direction of propagation of the electron beam. Equations (7) and (8) can be combined and yield on linearization

$$\frac{d^2\boldsymbol{a}_\perp}{dz^2} + 2i\,\frac{\omega}{c}\,\frac{d\boldsymbol{a}_\perp}{dz} = i\,\frac{\omega}{c^4}\,\boldsymbol{j}_\perp. \tag{16}$$

Similarly, (9) becomes

$$\left(i\,\frac{\omega}{c} + \frac{d}{dz}\right)a_z = -\frac{n}{c^2}\,. \tag{17}$$

Assume now that various frequencies characteristic to the electron beam (such as the plasma frequency ω_p and the cyclotron frequency $\Omega = e\mathcal{B}_0/mc$) are much less than ω. Then we expect $\boldsymbol{j}_\perp$, n, $\boldsymbol{a}$, and $\boldsymbol{b}$ to vary on the scale much longer than ω/c, or more precisely in order of magnitude for $X = j_\perp, n, a, b$

$$\left|\frac{d\ln X}{dz}\right| << \frac{\omega}{c}\,. \tag{18}$$

This disparity in scales allows us to simplify (16) and (17) significantly and rewrite them in the following approximate form:

$$\frac{d\boldsymbol{a}_\perp}{dz} = \frac{\boldsymbol{j}_\perp}{2c^3} \tag{19}$$

$$a_z = \frac{in}{\omega c}\,. \tag{20}$$

These are the desired field equations, describing the electromagnetic wave propagating along the amplifier.

III. Perturbed Current and Electron Density

At this stage we adopt the kinetic description of the electron beam, introduce the electron momentum distribution function $f(\boldsymbol{p}, z, t)$, and employ the Vlasov equation

$$\frac{\partial f}{\partial t} + v_z\,\frac{\partial f}{\partial z} - e\left[\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times(\boldsymbol{B}+\boldsymbol{\mathcal{B}})\right]\cdot\frac{\partial f}{\partial \boldsymbol{p}} = 0. \tag{21}$$

Choosing the cylindrical coordinate system $(\hat{e}_\perp, \hat{e}_z, \hat{e}_\phi)$ in the p-space and writing $f = f_0(p_z, p_\perp, \phi, z) + f_1(p_z, p_\perp, \phi, z, t)$ where f_1 is the perturbed part of the distribution caused by the electromagnetic wave we get in the zero order

$$\frac{\partial f_0}{\partial z} + \chi\,\frac{\partial f_0}{\partial \phi} = 0 \tag{22}$$

with

$$\chi = \frac{\Omega}{\gamma v_z}\,. \tag{23}$$

If initially (at $z = 0$) we have

$$f_0(p_z, p_\perp, \phi, 0) = \frac{m}{4\pi e^2}\,G(p_z, p_\perp, \phi) \tag{24}$$

then (22) yields

$$f_0(p_z, p_\perp, \phi, z) = \frac{m}{4\pi e^2}\,G(p_z, p_\perp, \phi - \chi z). \tag{25}$$

Consider now the first-order linearized Vlasov equation

$$\frac{\partial f_1}{\partial t} + v_z\,\frac{\partial f_1}{\partial z} - \frac{e}{c}\,(\boldsymbol{v}\times\boldsymbol{\mathcal{B}})\cdot\frac{\partial f_1}{\partial \boldsymbol{p}} = e\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\frac{\partial f_0}{\partial \boldsymbol{p}}\,. \tag{26}$$

Similarly to (11)-(14) let

$$f_1 = \mathrm{Re}\left[\frac{m}{4\pi e^2}\,\Psi(p_z, p_\perp, \phi, z)\,\Phi\right]. \tag{27}$$

Then (26) becomes

$$\left[-i\omega\left(1 - \frac{v_z}{c}\right) + v_z\left(\frac{\partial}{\partial z} + \chi\,\frac{\partial}{\partial \phi}\right)\right]\Psi = 4\pi e^2 c^2\left(\boldsymbol{a} + \frac{\boldsymbol{v}\times\boldsymbol{b}}{c}\right)\cdot\frac{\partial f_0}{\partial \boldsymbol{p}}\,. \tag{28}$$

Expressing $\boldsymbol{b}$ through $\boldsymbol{a}$ from (8), it can be shown that in (28) we can approximate

$$\boldsymbol{a} + \frac{\boldsymbol{v}\times\boldsymbol{b}}{c} = \frac{i}{\omega}\left[-i\omega\left(1 - \frac{v_z}{c}\right) + v_z\,\frac{\partial}{\partial z}\right]\boldsymbol{a}_\perp + \left(\frac{\boldsymbol{v}_\perp}{c}\cdot\boldsymbol{a} + a_z\right)\hat{e}_z. \tag{29}$$

It is convenient now to introduce the following orthonormal set of base vectors

$$\hat{e}_+ = \frac{1}{\sqrt{2}}\,(\hat{e}_y - i\hat{e}_x)$$

$$\hat{e}_- = \frac{1}{\sqrt{2}}\,(\hat{e}_y + i\hat{e}_x) \tag{30}$$

Then, on writing $\boldsymbol{a}_\perp = a_+\hat{e}_+ + a_-\hat{e}_-$, substituting (29) into (28), using (25) and expanding

$$\Psi = \sum_{n=-\infty}^{+\infty} A_n e^{in\phi} \tag{31}$$

$$f_0 = \frac{m}{4\pi e^2}\sum_{n=-\infty}^{+\infty} C_n e^{in\phi}, \quad G = \sum_{n=-\infty}^{+\infty} G_n e^{in\phi}. \tag{32}$$

Note that the coefficients A_0 and $A_{\pm 1}$ are the only ones necessary to know in order to find the perturbed current and density in the field equations (19) and (20). Indeed,

$$n = \iiint \Psi p_\perp\,dp_\perp\,dp_z\,d\phi = 2\pi\iint A_0 p_\perp\,dp_\perp\,dp_z \tag{33}$$

and

$$j_\perp = \iiint v_\perp \Psi p_\perp \, dp_\perp \, dp_z \, d\phi = \hat{e}_+ j_+ + \hat{e}_- j_- \tag{34}$$

where

$$j_\pm = \pm\sqrt{2}\pi i \iint v_\perp A_{\pm 1} p_\perp \, dp_\perp \, dp_z. \tag{35}$$

On Solving (29) for A_0 and $A_{\pm 1}$, substituting the solutions into (33) and (35), and integrating in the resulting equations by parts in order to eliminate the derivatives of the coefficients C_n with respect to $p_\perp$ and p_z, we finally get

$$n = 2\pi m c^2 \iint dp_\perp \, dp_z \left(\int_0^z dz' \, e^{-\alpha_0 \Delta z} \left\{ a_z C_0 \frac{v_\perp}{v_z^2}\left(1 - \frac{v_z^2}{c^2}\right)\left(1 + \frac{i\omega\Delta z}{v_z}\right) + \frac{i v_\perp^2}{\sqrt{2} c v_z^2} (a_- C_{+1} - a_+ C_{-1}) \cdot \left[1 + \frac{i\omega\Delta z}{v_z}\left(1 - \frac{v_z^2}{c^2}\right)\right] - \frac{\Omega \Delta z v_\perp^2}{2c^2 \gamma v_z^2} (a_- C_{+1} + a_+ C_{-1}) \right\} \right) \tag{36}$$

$$j_\pm = \pm\sqrt{2}\pi i m c^2 \iint dp_\perp \, dp_z \left(\frac{v_\perp}{\sqrt{2}\omega} \left[\mp 2 a_\pm C_0 - \frac{v_\perp^2}{c^2} (a_- C_{\substack{2\\0}} - a_+ C_{\substack{0\\-2}}) \right] + \int_0^z dz' \, e^{-\alpha_\pm \Delta z} \left\{ a_z C_{\pm 1} \frac{v_\perp^2}{v_z^2} \cdot \left[1 + \frac{i\Delta z}{v_z}\left(\omega \left\{ 1 - \frac{v_z^2}{c^2} \right\} \mp \frac{\Omega}{\gamma} \right) \right] + \frac{i v_\perp^3}{\sqrt{2} c v_z^2} \left[(a_- C_{\substack{2\\0}} - a_+ C_{\substack{0\\-2}}) \left(1 + \frac{v_z^2}{c^2} + \frac{i\Delta z}{v_z} \left\{ \omega \left(1 - \frac{v_z^2}{c^2} \right) \mp \frac{\Omega}{\gamma} \right\} \right) - (a_- C_{\substack{2\\0}} + a_+ C_{\substack{0\\-2}}) \frac{\Omega v_z}{\omega \gamma c} \left\{ 1 - \frac{i \Delta z \omega}{v_z} \right\} \right] + \frac{\sqrt{2} i \Omega v_\perp}{\gamma \omega v_z} a_\pm C_0 \right\} \right) + K_\pm \tag{37}$$

where $\Delta z = z - z'$ and the constants $K_\pm$ are chosen so that $j_\pm|_{z=0} = 0$, and

$$\alpha_0 = -\frac{i\omega}{v_z}\left(1 - \frac{v_z}{c}\right) \tag{38}$$

and

$$\alpha_\pm = \alpha_0 \pm i\chi. \tag{39}$$

According to (25) and (32)

$$C_n = G_n e^{-inxz}. \tag{40}$$

Note that the last expression for C_n, after being substituted into (36) and (37), allows one to express n and $j_\pm$ through the distribution function of the electrons at $z = 0$. At this point, we restrict our analysis to distribution functions of the form

$$G(p_z, p_\perp, \phi) = \frac{\omega_p^2}{2\pi p_\perp} \delta(p_z - p_{z0}) \, \delta(p_\perp - p_{\perp 0}) \, g(\phi). \tag{41}$$

In this case

$$C_n = \frac{\omega_p^2}{2\pi p_\perp} \delta(p_z - p_{z0}) \, \delta(p_\perp - p_{\perp 0}) \, g_n e^{-in\chi z} \tag{42}$$

where $g_n = [\int_0^{2\pi} g \exp(-in\phi) \, d\phi]/2\pi$. Thus, after performing the integration with respect to $p_\perp$ and p_z in (36) and (37), we get

$$n = R_0 + \int_0^z dz' \, \Delta z e^{-\alpha_0^0 \Delta z} S_0(z') + \int_0^z dz' \, e^{-\alpha_0^0 \Delta z} Q_0(z') \tag{43}$$

$$j'_\pm = R_\pm + \int_0^z dz' \, \Delta z e^{-\alpha_0^0 \Delta z} S_\pm(z') + \int_0^z dz' \, e^{-\alpha_0^0 \Delta z} Q_\pm(z') \tag{44}$$

where $j'_\pm = j_\pm e^{\pm i\chi_0 z}$, $\chi_0 = \Omega/\gamma_0 v_{z0}$, $\alpha_0^0 = -i(\omega/v_{z0})(1 - v_{z0}/c)$, $1/\gamma_0^2 = 1 - v_{z0}^2/c^2 - v_{\perp 0}^2/c^2$, and if we define $a'_\pm = a_\pm e^{\pm i\chi_0 z}$, then

$$R_0 = 0 \tag{45}$$

$$S_0 = \frac{i\omega\omega_p^2 c^2}{v_{z0}^3 \gamma_0} \left\{ \left(1 - \frac{v_{z0}^2}{c^2}\right) \left[a_z g_0 + \frac{i v_{\perp 0}}{\sqrt{2} c} (a'_- g_{+1} - a'_+ g_{-1}) \right] + \frac{i \Omega v_{z0} v_{\perp 0}}{\sqrt{2} \omega \gamma_0 c^2} (a'_- g_{+1} + a'_+ g_{-1}) \right\} \tag{46}$$

$$Q_0 = \frac{c^2 \omega_p^2}{\gamma_0 v_{z0}^2} \left[a_z g_0 \left(1 - \frac{v_{z0}^2}{c^2}\right) + \frac{i v_{\perp 0}}{\sqrt{2} c} (a'_- g_{+1} - a'_+ g_{-1}) \right] \tag{47}$$

$$R_\pm = \frac{i c^2 \omega_p^2}{\gamma_0 \omega} \left[-a'_\pm g_0 \mp \frac{v_{\perp 0}^2}{2c^2} (a'_- g_{\substack{2\\0}} - a'_+ g_{\substack{0\\-2}}) \right] + K_\pm e^{\pm i\chi_0 z} \tag{48}$$

$$S_\pm = \pm \frac{i c^2 v_{\perp 0} \omega_p^2}{\sqrt{2} \gamma_0 v_{z0}^3} \left\{ i a_z g_{\pm 1} \left[\omega \left(1 - \frac{v_{z0}^2}{c^2}\right) \mp \frac{\Omega}{\gamma_0} \right] - \frac{v_{\perp 0}}{\sqrt{2} c} (a'_- g_{\substack{2\\0}} - a'_+ g_{\substack{0\\-2}}) \left[\omega \left(1 - \frac{v_{z0}^2}{c^2}\right) \mp \frac{\Omega}{\gamma_0} \right] - \frac{v_{\perp 0} \Omega v_{z0}}{\sqrt{2} c^2 \omega \gamma_0} (a'_- g_{\substack{2\\0}} + a'_+ g_{\substack{0\\-2}}) \right\}, \tag{49}$$

$$Q_\pm = \frac{i c^2 \omega_p^2}{\sqrt{2} \gamma_0} \left\{ \pm \frac{v_{\perp 0}}{v_{z0}^2} \left[a_z g_{\pm 1} + \frac{i v_{\perp 0}}{\sqrt{2} c} \left(1 + \frac{v_{z0}^2}{c^2}\right) (a'_- g_{\substack{2\\0}} - a'_+ g_{\substack{0\\-2}}) - \frac{i v_{\perp 0} \Omega v_{z0}}{\sqrt{2} c^2 \omega \gamma_0} (a'_- g_{\substack{2\\0}} + a'_+ g_{\substack{0\\-2}}) \right] \pm \frac{i\sqrt{2}\Omega}{\gamma_0 \omega v_{z0}} g_0 a'_\pm \right\}. \tag{50}$$

Finally, we substitute (43) and (44) into the field equations (19) and (20) and we get the following complete set of integro-differential equations describing the evolution of the electromagnetic wave along the amplifier.

$$\frac{da'_\pm}{dz} \mp i\chi_0 a'_\pm = \frac{1}{2c^3} \cdot \left\{R_\pm + \int_0^z dz'\, e^{-\alpha_0^0 \Delta z}[\Delta z S_\pm(z') + Q(z')]\right\} \tag{51}$$

$$a_z = \frac{i}{c\omega}\left\{\int_0^z dz'\, e^{-\alpha_0^0\Delta z}[\Delta z S_0(z') + Q_0(z')]\right\}. \tag{52}$$

IV. Analysis of the Field Equations

Equations (51) and (52) comprise a set of linear integro-differential equations with all integrals in the form of convolutions. Thus, we can solve this system by means of the Laplace transformation. Namely, if we define for $\xi = \xi(z)$

$$\xi_k = \int_0^\infty e^{-ikz}\xi(z)\, dz \tag{53}$$

where Im k is assumed to be negative enough to assure convergence, we can apply transformation (53) to the field equations and get

$$i(k \mp \chi_0)\, a'_{\pm k} = \frac{1}{2c^3}\left[R_{\pm k} + \frac{S_{\pm k}}{(\alpha_0^0 + ik)^2} + \frac{Q_{\pm k}}{(\alpha_0^0 + ik)}\right] + a'_\pm(0) \tag{54}$$

$$a_{zk} = \frac{i}{c\omega}\left[\frac{S_{0k}}{(\alpha_0^0 + ik)^2} + \frac{Q_{0k}}{(\alpha_0^0 + ik)}\right]. \tag{55}$$

On using the expressions for the transforms S_{0k}, Q_{0k}, $R_{\pm k}$, $S_{\pm k}$, and $Q_{\pm k}$, we can rewrite (54) and (55) in the following vector form.

$$\begin{pmatrix} \epsilon_{++} & \epsilon_{+-} & \epsilon_{+z} \\ \epsilon_{-+} & \epsilon_{--} & \epsilon_{-z} \\ \epsilon_{z+} & \epsilon_{z-} & \epsilon_{zz}\end{pmatrix}\begin{pmatrix} a'_{+k} \\ a'_{-k} \\ a_{zk}\end{pmatrix} = \begin{pmatrix} a'_+(0) \\ a'_-(0) \\ 0\end{pmatrix} \tag{56}$$

where the dielectric tensor $\underline{\underline{\epsilon}}$ is given by

$$\epsilon_{\pm\pm} = i\left[k \mp \chi_0 + \frac{\omega_p^2 g_0}{2c\omega\gamma_0}\left(1 \mp \frac{\Omega}{\gamma_0\Delta} - Z_\pm\right)\right] \tag{57}$$

$$\epsilon_{\pm\mp} = -\frac{i\omega_p^2 v_{\perp 0}^2 g_{\pm 2}}{2c^2\gamma_0\Delta^2}\left[k \mp \chi_0\left(1 - \frac{v_{z0}}{c}\right)\right] \tag{58}$$

$$\epsilon_{\substack{+z\\-z}} = \mp \frac{v_{\perp 0}\omega_p^2 g_{\pm 1}}{2\sqrt{2}c\gamma_0\Delta^2}\left[\frac{\omega}{c}\left(1 - \frac{v_{z0}}{c}\right) + k \mp \chi_0\right] \tag{59}$$

$$\epsilon_{\substack{z+\\z-}} = \pm \frac{iv_{\perp 0}\,\omega_p^2 g_{\mp 1}}{\sqrt{2}c\gamma_0\Delta^2}\left[\left(1 - \frac{v_{z0}}{c} + \frac{ck}{\omega}\right) \mp \frac{\chi_0 v_{z0}}{\omega}\right] \tag{60}$$

$$\epsilon_{zz} = 1 - \frac{\omega_p^2}{\omega\gamma_0\Delta^2}\left(1 - \frac{v_{z0}^2}{c^2}\right)(\omega + ck) \tag{61}$$

and

$$\Delta = -\omega\left(1 - \frac{v_{z0}}{c}\right) + kv_{z0} \tag{62}$$

$$Z_\pm = -\frac{v_{\perp 0}^2\omega}{c\Delta^2}(k \mp \chi_0). \tag{63}$$

The dispersion relation is now defined via Det $\underline{\underline{\epsilon}} = 0$. The knowledge of the roots $k = k(\omega)$ of this dispersion relation allows one, in principle, to apply the inverse Laplace transformation to the solution $\boldsymbol{a}_k$ of (56) and thus find the actual z-dependence of the amplitude of the wave along the amplifier. Nevertheless, because of the complexity of the dielectric tensor $\underline{\underline{\epsilon}}$, the inversion of the transforms in our case is a rather complicated algebraic procedure. Usually in such situations one restricts the study to the search of the roots of the dispersion relation only, which allows one to find the asymptotic z-dependence of spatially unstable modes. We use this approach in the next section and find roots of the dispersion relation for several configurations of electron beams. In addition, in order to avoid complexity of taking the inverse Laplace transformation, but nevertheless willing to find the z-dependence of the fields, we solve the field equations directly in the next section. With this purpose in mind, we transform here the field equations (51) and (52) into a system of first order ordinary differential equations.

Define

$$I_1^\alpha = \int_0^z dz'\, e^{-\alpha_0^0\Delta z} S_\alpha(z') \tag{64}$$

$$I_2^\alpha = \int_0^z dz'\, e^{-\alpha_0^0\Delta z}[Q_\alpha(z') - z' S_\alpha(z')] \tag{65}$$

where α is +, - or 0. Then (51) and (52) can be rewritten as

$$\frac{da'_\pm}{dz} \mp i\chi_0 a'_\pm = \frac{1}{2c^3}(R_\pm + zI_1^\pm + I_2^\pm) \tag{66}$$

$$a_z = \frac{i}{c\omega}(zI_1^0 + I_2^0), \tag{67}$$

and, differentiating (64) and (65),

$$\frac{dI_1^\alpha}{dz} = -\alpha_0^0 I_1^\alpha + S_\alpha \tag{68}$$

$$\frac{dI_2^\alpha}{dz} = -\alpha_0^0 I_2^\alpha + Q_\alpha - zS_\alpha. \tag{69}$$

Equations (66)-(69) comprise a complete set of first order differential equations, which can be solved numerically with an appropriate set of initial conditions.

V. Stability Analysis and Direct Solution of the Field Equations

In this section we apply the theory to the following three electron momentum distribution functions.

1) A randomly gyrophased electron beam, characterized by $g(\phi) = 1$, so that $g_0 = 1$ and $g_n = 0$ ($n = \pm 1, \pm 2, \cdots$).

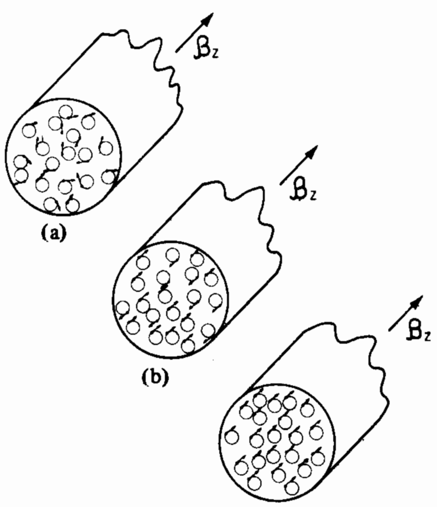

Fig. 1. Schematic of transverse cross sections of various beam configurations. (a) Randomly gyrophased beam, (b) double helical beam, (c) helical beam. The arrows show the directions of the transverse velocities of individual electrons in the beam.

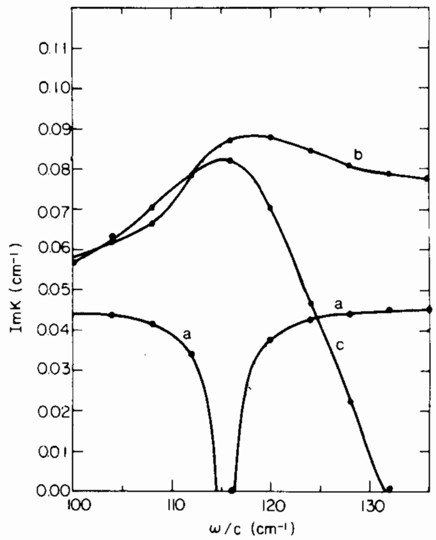

Fig. 2. Spatial growth rates Im k versus normalized frequency ω/c. The parameters are $\omega_p^2/c^2 = 2$ cm^{-2}, $\chi_0 = 3$ cm^{-1}, $\gamma_0 = 5$, and $v_{\perp 0}/c = 0.1$. (a) Randomly gyrophased beam, (b) double helical beam, (c) helical beam.

2) A double helical beam with $g(\phi) = \pi\,[\delta(\phi - 0) + \delta(\phi - \pi)]$. In this case $g_{2m} = 1$ and $g_{2m+1} = 0$ $(m = 0, \pm 1, \pm 2, \cdots)$.

3) A helical beam, described by $g(\phi) = 2\pi\delta\,(\phi - 0)$ and $g_n = 1$ $(n = 0, \pm 1, \pm 2, \cdots)$. These three types of the electron beam are illustrated schematically in Fig. 1.

In the case of an azimuthally symmetric electron beam [case 1)] all the off-diagonal elements of the dielectric tensor $\underline{\underline{\epsilon}}$ vanish and the dispersion relation simply becomes $\epsilon_{++}\epsilon_{--}\epsilon_{zz} = 0$. In this case the three possibilities ϵ_{++}, ϵ_{--}, and $\epsilon_{zz} = 0$ correspond respectively to the right-hand transverse wave, the left-hand transverse wave, and the relativistic longitudinal space-charge wave. The equation $\epsilon_{++} = 0$ is identical to the dispersion relation derived in [9] for the case $\omega/c \gg \chi_0$. As was shown in [9], $\epsilon_{++} = 0$ yields for Δ small enough and large ω a pair of complex roots for k, one of which has a negative imaginary part

$$\mathrm{Im}\,k \simeq -\frac{\omega_p}{\sqrt{2\gamma_0}}\,\frac{v_{\perp 0}}{c v_{z0}} \tag{70}$$

and therefore describes a spatially unstable mode in the amplifier.

We now consider cases 2) and 3). In case 2) the dispersion relation is given by

$$(\epsilon_{++}\epsilon_{--} - \epsilon_{+-}\epsilon_{-+})\,\epsilon_{zz} = 0. \tag{71}$$

We see that the left-hand and right-hand modes are coupled. Nevertheless, as in case 1) the space-charge mode is still uncoupled. The reason for this is that both distributions 1) and 2) are azimuthally symmetric and, therefore, the average unperturbed transverse velocity in the beam $\langle v_{\perp 0}\rangle$ is zero (see Section I).

In contrast to cases 1) and 2), the distribution function of the electrons in case 3) is azimuthally asymmetric, and as a result all the off-diagonal elements of the dielectric tensor are nonzero. In this case the space-charge mode couples to the transverse modes. Because of the complexity of the dispersion relation in cases 2) and 3), their analytic study becomes difficult. We therefore find the roots numerically for the sample case: $\omega_p^2/c^2 = 2$ cm^{-1}, $\chi_0 = 3$ cm^{-1}, $\gamma_0 = 5$, and $v_{\perp 0}/c = 0.1$. This set of parameters is typical of a collective type Raman free electron laser. In Fig. 2 we compare the computed growth rates for the three distribution functions 1), 2), and 3). The solid lines represent the solutions of the dispersion relation for the sample case, and the dots were found by solving the field equations directly for large values of z, where the exponentially growing modes with the largest growth rates are dominant. We see from the figure that in case 1) (the randomly phased electron beam) the maximum growth rate is 0.044 cm^{-1} in agreement with (70). For the double helical beam [case 2)] the growth rate at maximum is 0.087 cm^{-1}. The growth rates found for the helical beam [case 3)] agree well with the results of the cold fluid theory [10] and for both cases 2) and 3) are comparable in magnitude with the growth rates one has in conventional FEL's operating in similar regimes [11]. Thus, we see in Fig. 2 that the coupling between the transverse modes in case 2) enhances the gain. In case 3) the enhancement effect comes from the coupling to the space-charge modes, which enables the axial density bunching to drive the instability.

The improved operation of the amplifier in the cases of the helical and double-helical beams is demonstrated in Fig. 3, where the actual z-dependence of the gain along the amplifier is shown for aforementioned three distributions in the sample case. These results were obtained by solving the field equations (66)-(69) directly. We see in the figure that the exponential growth for distribution 1) becomes dominant only after the beam passes 60-100 cm along the device, while in the cases of the helical and double-helical beams the growth is exponential already at ~30 cm and its actual value quickly becomes very high.

These results should motivate attempts to generate helical beams for a practical nonwiggler FEL. One way is to shoot

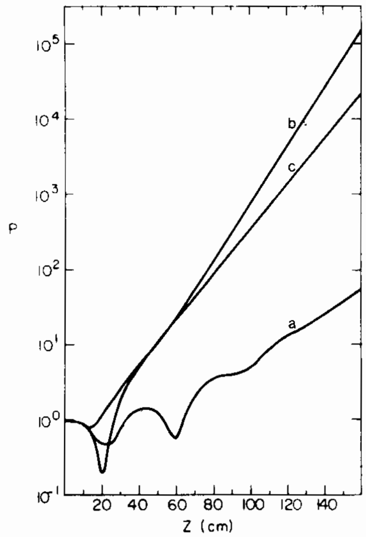

Fig. 3. The square root of the relative power gain $P = |a(z)|/|a(0)|$ versus the interaction distance in the sample case for $\omega/c = 120\ \mathrm{cm}^{-1}$. (a) Randomly gyrophased beam, (b) double helical beam, (c) helical beam.

the beam at an angle to the magnetic field. Perhaps more promising is to pass the beam through a magnetic "kicker" which will give all the electrons the same perpendicular momentum component.

In conclusion,

1) We have presented a kinetic theory of nonwiggler FEL operating in strong uniform guide magnetic fields. The amplifier problem is reduced to a solution of a system of first order ordinary differential equations for the electric component of the electromagnetic field.

2) Our numerical examples demonstrate the potential of operating a nonwiggler FEL in the collective regime, where the spatial growth rates can be comparable to those in the convential FEL's.

3) The form of the azimuthal distribution of the momentum of the electrons in the beam in the nonwiggler FEL is extremely important and influences the growth rates in both their magnitude and form. The asymmetry in the azimuthal distribution results in higher gains in the system due to the coupling of the transverse and space-charge modes.

ACKNOWLEDGMENT

The authors would like to thank Prof. J. L. Hirshfield, Prof. P. Avivi, and Dr. H. Mitchell for their helpful comments and suggestions in the preparation of this paper.

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