Inequalities for moduli of smoothness versus embeddings of function spaces

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Define on $L^p(\mathbb{R}^n)$, $p \geq 1$, moduli of smoothness of order r, $r \in \mathbb{N}$, by

$$\omega_r(t,f)_p := \sup_{|h|<t} \|\Delta_h^r f\|_p\,, \quad t>0,\ \ \Delta_h f(\cdot) = f(\cdot + h) - f(\cdot),\ \Delta_h^r = \Delta_h \Delta_h^{r-1}.$$

Trivially one has $\omega_r(t,f)_p \lesssim \omega_k(t,f)_p\,,\ k < r$. Its converse is known as Marchaud inequality. M.F. Timan 1958 proved a sharpening of the converse, nowadays called *sharp Marchaud inequality*, which in the present context takes the form,

$$\omega_k(t,f)_p \lesssim t^k \left(\int_t^\infty [s^{-k}\omega_r(u,f)_p]^q \frac{du}{u} \right)^{1/q}, \qquad t>0, \quad k<r.$$

where $q := \min(p,2)$, $1 < p < \infty$. Here we will show that the sharp Marchaud inequality as well as further sharp inequalities for moduli of smoothness like Ulyanov and Kolyada type ones are equivalent to (known) embeddings between Besov and potential spaces.
To this end one has to make use of moduli of smoothness of fractional order which can be characterized by Peetre's (modified) K-functional, living on L^p and associated Riesz potential spaces. Limit cases of the Holmstedt formula (connecting different K-functionals) show that the embeddings imply the desired inequalities. Conversely, the embeddings result from the inequalities for moduli of smoothness by limit procedures.