Universality in spectral statistics of open quantum graphs

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Outline

- Edge spectral universality for strongly open quantum graphs
- Supersymmetry versus Semiclassical approach
- Quantum maps. Edge fractional Weyl law
- Combinatorial applications
Quantum graphs

Edges: \( \psi_i = A_i e^{ikx} + B_i e^{-ikx} \),

\( \ell_i \) - lengths of edges \( i = 1, \ldots B \)

Vertices: \( A_l = \sum_k \sigma_{l,k}^{(v)} B_k \)

Spectrum is real, determined by secular equation:

\[
\det(I - S(k)) = 0, \quad S(k) = \Sigma \Lambda_k
\]

\( \Sigma \) is \( 2B \times 2B \) unitary matrix composed of \( \sigma^{(v)} \)

\[
\Lambda_k = \text{diag}(e^{ik\ell_1}, e^{ik\ell_2}, \ldots e^{ik\ell_N}), \quad N = 2B
\]
Open quantum graphs

Spectrum is complex, still determined by:

$$\det(I - S(k)) = 0, \quad S(k) = \sum \Lambda_k$$

but $$\sum$$ (equiv. $$S(k)$$) is non-unitary

**Motivation:** Model for chaotic scattering. Experiments. Combinatorics applications. etc.
Strongly open quantum graphs

Weakly open system: # of open channels is finite

Strongly open system: # of open channels $\sim N \implies$ grows in the semiclassical limit as the size of the system

We will consider the later case
Spectral universality in closed graphs

Spectral density:

\[ S(k)|m\rangle = \lambda_m|m\rangle \]

\[ \rho(\theta) = \frac{1}{N} \sum_{m=1}^{N} (\theta - \theta_m), \quad \lambda_m = e^{i\theta_m} \]

Spectral correlations in closed quantum graphs are universal. T. Kottos U. Smilansky (1997)

If time reversal symmetry broken (\(\Sigma \neq \Sigma^*\)) \(\implies\) the same as in circular unitary ensemble (CUE).

Reason: Classical dynamics are “chaotic” (Bohigas-Giannoni-Schmit conjecture)
Spectral universality in closed graphs

**Example:** Two point correlation function

\[
\langle \rho(\theta)\rho(\theta + 2\pi \varepsilon/N) \rangle = \text{F.T.} [K(\tau)]
\]

The same as for CUE:

\[
K(\tau) := \frac{1}{N} \left\langle \left| \sum_{m=1}^{N} \lambda_m^n \right|^2 \right\rangle = \begin{cases} 
\tau & \text{if } \tau \leq 1 \\
0 & \text{if } \tau > 1 \end{cases}, \quad \tau = n/N
\]

Holds if the gap in the spectrum \( \{\mu_k\} \) of classical evolution \( |\Sigma_{i,j}|^2 \) is large enough S. Gnutzmann, A. Altland (2004); G. Tanner (2001):

\[
\mu_1 = 1 \geq \mu_2 \geq \ldots \mu_N, \quad 1 - \mu_2 \sim N^{-\kappa}, \quad \kappa < 1
\]
Spectral universality in open graphs

Radial spectral density:

\[ \hat{\rho}(r) = \frac{1}{N} \sum_{m=1}^{N} (r - |\lambda_m|) \]

Does any form of universality holds for open quantum graphs?

\[ \rho(r) := \langle \hat{\rho}(r) \rangle = ? \]
\[ K(\tau) := \frac{1}{N} \left\langle \left| \sum_{m=1}^{N} \lambda_m^n \right|^2 \right\rangle = ? \]
Main result
Spectral Universality


1) Spectral edge of \( S = \sum \Lambda_k \) is at \( r = \sqrt{\mu_1} \), where \( \mu_1 \) is the largest eigenvalue of \( Q \), \( Q_{i,j} = |\Sigma_{i,j}|^2 \)

2) Spectral density & correlations of matrices \( \frac{1}{\sqrt{\mu_1}} \sum \Lambda_k \) are universal at \( 1/\sqrt{N} \) neighborhood of the edge

**Necessary conditions:**

**(A)** Large gap in the spectrum of \( |\Sigma_{i,j}|^2 \),

\[ \mu_1 - \mu_2 = O( N^{-\kappa} ) , \quad \kappa < 1/2 \]

*Example:* Random regular graph, \( \Sigma_{i,j} \) – connectivity matrix, Friedman (2003).

**(B)** “Strong” non-unitarity of \( \Sigma \)
Density of states

Spectral density:
\[ \rho(1 + s/\sqrt{N}) = \frac{1}{\mu} \text{erfc} \left( \frac{s}{\sqrt{2\mu}} \right) + O \left( \frac{1}{\sqrt{N}} \right) \]

At the edge:
\[ \rho(1) = 1/\mu + O(1/\sqrt{N}), \quad \mu \text{ depends on } \Sigma \]
Form factor

Relevant scaling \( n \sim \sqrt{N} \to \infty, \ t = n/\sqrt{N} \)

\[
\frac{1}{n} \langle |\Tr S^n|^2 \rangle = \frac{2}{\mu t^2} \sinh(\mu t^2/2) + O(1/\sqrt{N}) =
\]

\[
= \frac{1}{\mu t^2} \exp(\mu t^2/2) - \frac{1}{\mu t^2} \exp(-\mu t^2/2) + O(1/\sqrt{N})
\]

\[
\text{diagonal} \quad \text{correlations}
\]

Comes from

\[
\text{Diagonal} : \langle |\lambda_i|^{2n} \rangle \quad \& \quad \text{Correlations} : \langle \lambda_i^n \lambda_j^n \rangle
\]

\[
\int \rho(r)r^{2n}dr
\]
Comparison with RMT

Same result for some RMT ensembles e.g, truncated CUE:

\[
U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{N \times N}, \quad U \in \text{CUE}
\]


\[
\rho \left( \frac{1}{\sqrt{2}} + \frac{s}{\sqrt{N}} \right) \sim \text{erfc} \left( \sqrt{2} s \right)
\]

Same asymptotics for Ginibre, but with different scaling
Supersymmetry approach

$$\rho(r) = \frac{1}{N \pi^2 r} \lim_{\varepsilon \to 0} \Im \partial_{\theta_1} \partial_{\theta_2} Z(\varepsilon, \theta_1, \theta_2) \bigg|_{\theta_{1,2}=0}$$

Generating function:

$$Z(\varepsilon, \theta_1, \theta_2) = \left\langle \frac{\xi(\theta_1 + \varepsilon) \xi^*(\theta_2 - \varepsilon)}{\xi(-\theta_1 + \varepsilon) \xi^*(-\theta_2 - \varepsilon)} \right\rangle_k$$

$$\xi(\theta) = \det \left( I - \frac{e^{i\theta}}{r} \Sigma \Lambda_k \right), \Lambda_k = \text{diag}(e^{i\varphi_1}, \ldots e^{i\varphi_N}), \varphi_j = k \ell_j$$

Average over $k \Leftrightarrow$ Average over $\int \prod_{i=1}^{N} d\varphi_i$
3-step procedure

S. Gnutzmann, A. Altland (2004); Z. Pluhar, HA Weidenmüller (2013)

**Step I.** Gaussian integral representation:

\[
\frac{\det A_f}{\det A_b} = \int d[\tilde{\psi}, \psi] e^{-(\tilde{\psi}, A\psi)}
\]

\[
\psi = (z_1, \ldots, z_N, \eta_1, \ldots, \eta_N), \quad A = \begin{pmatrix} A_b & 0 \\ 0 & A_f \end{pmatrix}
\]

**Step II.** Colour-flavour transformation:

\[
\int \prod_{i=1}^N \frac{d\varphi_i}{2\pi} f(\varphi, \psi, \tilde{\psi}) \rightarrow \int d(Z, \tilde{Z}) F(Z, \tilde{Z}, \psi, \tilde{\psi})
\]

\[Z = \text{diag}(Z_1, \ldots, Z_N), \text{ each } Z_i - 2 \times 2 \text{ supermatrix}\]

**Step III.** Integrating out the Gaussian fields \(\tilde{\psi}, \psi\)
Large $N$ limit (Unitary case)

\[ Z(\varepsilon, \theta_1, \theta_2) = \int d(Z, \tilde{Z}) e^{-S(Z, \tilde{Z})} \]

\[ S = \text{str} \left( \log(1 - Z\tilde{Z}) - \log(1 - \frac{e^{i\varepsilon}}{r^2} Z\Sigma_{\theta_1} \tilde{Z}\Sigma_{\theta_2}^\dagger) \right) \]

\[ \Sigma_{\theta} = \Sigma \otimes \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \quad Z = \text{diag}(Z_1, \ldots Z_N) \]

**Unitary case:** $\Sigma\Sigma^\dagger = 1$, $r = 1$, the saddle-point manifold:

\[ Z_i = Y, \tilde{Z}_i = \tilde{Y}, \quad \tilde{Y}_{bb} = Y_{bb}^*, \tilde{Y}_{ff} = -Y_{ff}^*, |Y_{bb}|^2 \leq 1 \]

Integrating over $2 \times 2$ supermatrices $Y, \tilde{Y} \implies \text{GUE result}$
Large $N$ limit (Non-unitary case)

If $\Sigma$ is non-unitary $\implies$ the saddle-point: $Z = \tilde{Z} = 0$

$$S = \text{str} \left( \tilde{Z} (I - r^{-2}Q \otimes I_{2 \times 2}) Z \right) + O((Z\tilde{Z})^2), \quad Q_{i,j} = |\Sigma_{i,j}|^2$$

Nmodes:

\[
\begin{align*}
Z &= \sum Y_i \otimes \chi^{(i)}, \quad \chi^{(i)} = \text{diag}(\chi_1^{(i)}, \ldots, \chi_N^{(i)}) \\
\tilde{Z} &= \sum \tilde{Y}_i \otimes \tilde{\chi}^{(i)}, \quad \tilde{\chi}^{(i)} = \text{diag}(\tilde{\chi}_1^{(i)}, \ldots, \tilde{\chi}_N^{(i)})
\end{align*}
\]

$\chi^{(i)}, (\tilde{\chi}^{(i)})$- left (right) eigenvectors: $Q\chi^{(i)} = \mu_i \chi^{(i)}$

$$S = \sum_{i=1}^{N} \left( 1 - r^{-2}\mu_i \right) \text{str} \left( \tilde{Y}_i Y_i \right) + O((Z\tilde{Z})^2),$$

At $r^2 = \mu_1$ the highest mode is massless $\implies$ Others can be neglected if gap is large: $\mu_1 - \mu_2 = O(N^{-\kappa}), \kappa < 1/2$
Large $N$ limit (Non-unitary case)

Integration over massless mode $Y_1 = Y, \tilde{Y}_1 = \tilde{Y}$:

$$Z(\varepsilon, \theta_1, \theta_2) = \int d[Y, \tilde{Y}] \exp (-NS_{RMT})$$

$$S_{RMT} = \left(1 - \frac{e^{i\varepsilon}}{r^2}\right) \text{str} (\tilde{Y}Y) - \frac{\mu e^{i2\varepsilon}}{2r^4} \text{str} (\tilde{Y}(1 + i\theta_1\sigma)Y(1 + i\theta_2\sigma))^2$$

$$\downarrow$$

$$\rho(\sqrt{\mu_1} + r/\sqrt{N}) = \frac{1}{N\pi^2} \lim_{\varepsilon \to 0} \text{Im} \partial_{\theta_1} \partial_{\theta_2} Z(\varepsilon, \theta_1, \theta_2) \bigg|_{\theta_1,2=0}$$

$$= \frac{1}{\mu \sqrt{\mu_1}} \text{erfc} \left( \frac{r \sqrt{\mu_1}}{2\mu} \right) + O \left( N^{-1/2} \right)$$

Scaling parameter: $\mu = \frac{1}{N} \text{Tr} \left( (\Sigma \Sigma^\dagger \bar{\Sigma})^2 - (\Sigma \Sigma^\dagger)^2 \right)$
“Semiclassical” approach

\[ \text{Tr} S^n = \sum_{\Gamma} A_{\Gamma} \exp (i k l_{\Gamma}) \]

\( A_{\Gamma} = \sum_{i_1,i_2} \cdot \sum_{i_2,i_3} \cdots \sum_{i_n,i_1} \) - “Stability”

\( l_{\Gamma} \) - Length of \( \Gamma \)
\[
\langle |\mathrm{Tr}S^n|^2 \rangle = \sum_{\Gamma} |A_{\Gamma}|^2 + \sum_{\Gamma,\Gamma'} A_{\Gamma} A_{\Gamma}^*,
\]

\(D^{(0)}\) -diagonal

\[
D^{(0)} = \frac{1}{\mu^1_n} \sum_{i_1,\ldots,i_n} |\Sigma_{i_1,i_2}|^2 |\Sigma_{i_2,i_3}|^2 \ldots |\Sigma_{i_n,i_1}|^2 =
\]

\[
= \frac{1}{\mu^1_n} \mathrm{Tr}Q^n = 1 + O \left( N^{-\frac{1}{2}+\kappa} \right)
\]
Second order

\[
D^{(2)} = \begin{pmatrix}
\text{(Loops)} & \text{(Encounters)} & \text{(Structure)} \\
\frac{1}{4!} \left( \frac{n}{N} \right)^4 & (N \mu_N)^2 & 1
\end{pmatrix}
\]

\[
\mu_N = \frac{1}{N} \text{Tr} \left( (\Sigma \chi \Sigma^\dagger \bar{\chi})^2 - (\chi \bar{\chi})^2 \right)
\]

\[
\chi = \text{diag}(\chi_1, \ldots, \chi_N), \quad \bar{\chi} = \text{diag}(\bar{\chi}_1, \ldots, \bar{\chi}_N). \quad \chi_i, \bar{\chi}_i
\]

elements of highest left (right) eigenvector of \( \mathcal{Q} \), \((\chi, \bar{\chi}) = N\)
All orders

If $\chi = \bar{\chi} = (1, 1, \ldots 1)$ e.g., $Q$ is doubly stochastic matrix:

$$\mu_N = \frac{1}{N} \text{Tr} \left( \Sigma \Sigma^\dagger \right)^2 - 1$$

Higher orders:

$$D^{(k)} = \left( \frac{1}{(2k)!} \left( \frac{n}{2N} \right)^{2k} \right) \left( (N\mu_N)^k \right) \left( \frac{(2k)!}{k!} \right) + O \left( N^{-\frac{1}{2} + \kappa} \right)$$

If $\lim_{N \to \infty} \mu_N = \mu > 0$ (Non-unitarity condition B)

$$\frac{1}{n} \langle |\text{Tr} S^n|^2 \rangle = \sum_{\text{even } k} D^{(k)} = \frac{2N}{\mu n^2} \sinh \left( \frac{\mu n^2}{2N} \right) + O \left( N^{-\frac{1}{2} + \kappa} \right)$$

$\Rightarrow$ Density of states (the same result as by SUSY)
Consider inverse matrices:

\[ \Sigma^{-1} \Lambda^* \],  \quad \text{Spectrum: } \{ \lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_N^{-1} \} \]

The spectral density \( \rho'(r) = r^{-2} \rho(1/r) \Rightarrow \)

**The inner edge of ensemble \( \Sigma\Lambda\varphi \) related to the outer edge of \( \Sigma^{-1} \Lambda^* \)**

E.g., the position of the inner edge:

\[ r = \mu_1'^{-1} \]

\( \mu_1' \) is the largest eigenvalue of \( |\Sigma_{i,j}|^2 \). No inner edge if \( \Sigma^{-1} \) does not exist.
Implications for dynamical systems?
Open quantum maps

**Triadic baker’s map:**

\[ T \cdot (q, p) = \begin{cases} (3q, \frac{1}{3}p) & \text{if } q \in [0, \frac{1}{3}) \\ (3q - 1, \frac{1}{3}p + 1) & \text{if } q \in [\frac{1}{3}, \frac{2}{3}) \\ (3q - 2, \frac{1}{3}p + 2) & \text{if } q \in [\frac{2}{3}, 1) \end{cases} \]

\[
{x} = (q, p) = \ldots x_{-1} x_0 \cdot x_1 x_2 \ldots; \quad x_i \in \{0, 1, 2\}
\]

Quantisation:

\[
U_a = U \cdot A, \quad A = \text{diag}\{a_1, a_2 \ldots a_N\}, \quad a_i = a(i/N),
\]

\(a(x)\) is a smooth function on \([0, 1]\) (modeling absorption)

\(N \times N\)-matrix \(U\) - quantisation of \(T\)
Some known properties

1) **Full absorption**: \( a(x) = 0 \) for \( x \in [x_1, x_2] \) \( \implies \) Fractal Weyl law

W. T. Lu, S. Sridhar, M. Zworski, (2003): # of eigenstates \( |\lambda_i| \geq r_0 > 0 \), is \( \sim N^{d/2} \), \( d \)-fractal dimension of the classical repeller. E.g., triadic baker’s map:
\[
\mathcal{R} = \{ x = x_{-1}x_0 . x_1x_2x_3 . . . | x_i \in \{0, 1\} \}, \quad d = \frac{\log 4}{\log 3}
\]

2) **Partial absorption**: \( 0 < a(x) \) \( \implies \) for \( N \to \infty \) the eigenvalue distribution \( \rho(r) \) is concentrated at \( r = < \log a > \) with width \( \sim 1/\sqrt{\log N} \)

S. Nonnenmacher E. Schenck (2008)
Quantum graph as map quantisation

\[ \mathcal{T}(q, p) = \begin{cases} 
(2q, \frac{1}{2}p) & \text{if } q \in [0, \frac{1}{2}) \\
(2 - 2q, 1 - \frac{1}{2}p) & \text{if } q \in \left[\frac{1}{2}, 1\right) 
\end{cases} \]

Quantisation: \( U = \Lambda \phi U_0, \quad \Lambda \phi = \text{diag}\{e^{\phi_1}, \ldots, e^{\phi_N}\}, \quad N = 2^p \)
Walsh quantised open baker’s map

Quantum map: \( U_a = U \cdot A, \quad A = \text{diag}\{a_1, a_2 \ldots a_N\} \)

Classical evolution: \( Q_a = Q \cdot A^2, \quad Q_{i,j} = |U_{i,j}|^2 \)

Density of \( U_a \) eigenvalues \( \rho(\text{edge}) = \frac{1}{\mu_N} (1 + O(1/\sqrt{N})) \)

\[
\mu_N = \frac{1}{N} \text{Tr} \left( (U_a \chi U_a^\dagger \bar{\chi})^2 - (\chi \bar{\chi})^2 \right) \sim N^{d/2+1} \quad (?)
\]

\( \chi, \bar{\chi} \) - left/right eigenvectors of \( Q_a \) for the highest eigenvalue
**Edge fractal Weyl law**

The exponent \( d \) is determined by fractal dimensions of left/right equilibrium eigenmeasures:

\[
\frac{1}{N} \text{Tr}(\chi \bar{\chi})^2 = \frac{1}{N} \sum_{k=1}^{N} \chi_k^2 \bar{\chi}_k^2 \sim N^{d/2+1}
\]

**Conjecture:** The above is true for a generic quantisation

**Example** - triadic baker’s map with full absorption \( N = 3^p \):

\[
\chi = \underbrace{\omega \otimes \omega \ldots \otimes \omega}_p, \bar{\chi} = \underbrace{\bar{\omega} \otimes \bar{\omega} \ldots \otimes \bar{\omega}}_p, \omega = \sqrt{\frac{3}{2}}(1, 1, 0), \bar{\omega} = (1, 1, 1)
\]

\[\implies d = \log 4/\log 3, \rho(r) = C(r)N^{d/2+1}(1 + O(1/\sqrt{N})).\]

\( C(r) \) is universal only at the edge vicinity
Seven bridges of Königsberg

Original problem:
Whether a path exists?

If yes $\implies$ refine question:
How many Eulerian paths exist?

Motivation: Clusters of periodic orbits with close actions in chaotic systems
Clusters of periodic orbits

\[ Q \] - Edge connectivity matrix

\[ \Lambda(\phi) = \text{diag}(e^{i\phi_1}, e^{i\phi_2} \ldots e^{i\phi_N}) \]

\[ \text{Tr}(Q \Lambda\phi)^n = \sum_{\vec{n}} |C_{\vec{n}}| \exp(i(\vec{n}, \phi)), \quad (\vec{n}, \phi) = \sum n_k \phi_k \]

\[ \vec{n} = (n_1, \ldots n_N), \quad n_k - \text{number of passages through edge } k \]

Moments of cluster distribution

\[ Z_k = \sum_{\vec{n}} |C_{\vec{n}}|^k \iff \text{Traces of } Q \Lambda\phi \]
Second moment

\[ Z_2 = \langle |\text{Tr}(Q\Lambda \phi)^n|^2 \rangle_\phi = \langle | \sum_{i=1}^{N} \lambda_i^n |^2 \rangle_\phi \]

\( \lambda_i, \ i = 1, \ldots N \) are eigenvalues of non-unitary \( Q\Lambda \phi \)

Spectral form-factor for a non-unitary quantum graph

A. \( N \) is fixed, \( n \to \infty \) \( \implies \) Need to know the distribution of largest eigenvalue \( \lambda_{\text{max}}(\phi) \)

B. \( N, n \to \infty \) \( \implies \) Need to know the density and correlations of \( \lambda_i \)’s
Average cluster size

d-regular graph:

\[ < C > = \frac{\sum_i |C_i|^2}{n \sum_i |C_i|} = \frac{\sinh \nu}{\nu} \left( 1 + O \left( \frac{1}{\sqrt{N}} \right) \right) \]

\[ \nu = \frac{dn^2}{2N(d - 1)} \] - average number of encounters

♣ Short periodic orbits \( n \lesssim \sqrt{N} \). Most of the clusters contain just one periodic orbit.

♦ At \( n \sim \sqrt{N} \) starts exponential growth of cluster sizes

♣♣ At \( n \sim N \) (Heisenberg time) periodic orbits “condense” into huge clusters
Summary

Universality of spectral correlations $\iff$ Universality at the edges correlations & density

- Unitary: $\sim N^{-1}$
- Non-unitary: $\sim N^{-1/2}$

Saddle-point manifold: $\iff$ Simple saddle-point
All diagrams: $\iff$ 2-encounter diagrams

Transitional case (weak opening) $\mu_N \sim 1/N$: All diagrams must be included
Summary

Density of states & correlations depend on one scaling parameter:

$$\mu_N = \frac{1}{N} \text{Tr} \left( (\Sigma \mathcal{X} \Sigma^\dagger \bar{\mathcal{X}})^2 - (\mathcal{X} \bar{\mathcal{X}})^2 \right)$$

In particular:

$$\rho(\text{edge}) = 1/\mu_N(1 + O(1/\sqrt{N}))$$

The large $N$ behavior of $\mu_N$ depends on the type of graph i.e., $\Sigma$.

Q1. Do these results generalize to generic open quantum maps/Hamiltonian systems?
Q2. Distribution of eigenfunctions? Quantum ergodicity?